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Uniform Asymptotic Expansions

by

CLIVE CHESTER and BERNARD FRIEDMAN

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§1. Introduction

In many cases of practical importance there arises the problem of determining the asymptotic behavior for large values of n of a complex integral of the form

$$F(n,\alpha) = \int_C e^{-nf(z,\alpha)}\,\Phi(z)\,dz \tag{1.1}$$

where n is a positive real number, f and Φ are meromorphic functions of the complex variable z, f is continuous in the complex parameter α, $\Phi(z)$ is of at most exponential growth at infinity and C is a contour in the z-plane which extends to infinity in both directions and on which Re f > 0 sufficiently far out (to ensure convergence of the integral).

Some of the assumptions about n and C are not essential. If n is complex, but has a fixed argument θ, we only have to write $n = |n|e^{i\theta}$ and absorb the $e^{i\theta}$ into the function f. If C has one or both end points at finite points, then it will be seen that the following discussion need undergo only trivial alterations.

A method of attacking the problem has long been known and often utilized. We shall discuss this method in greater detail in the next section. In its simplest form, the method, often called "the saddle point method", consists in first making the change of variable

$$\zeta = f(z,\alpha) \tag{1.2}$$

in (1.1) and then showing that the asymptotic behavior

can be obtained by finding the branch points z_b of the mapping, i.e. the points where $f_z(z_b,\alpha) = 0$, taking their images, ζ_b, in the ζ-plane, expanding z and therewith $\overline{\phi}(z)\frac{dz}{d\zeta}$ into series in fractional powers of $(\zeta-\zeta_b)$, and integrating termwise. The results, as will be shown in the next section, are series of the form $e^{-n\zeta_b}$ times a power series in powers of $n^{-1/\nu}$ where ν is the order of the first non-vanishing derivative of f at z_b. Due to the presence of the exponential factor, it is clear that the branch point ζ_b with the smallest real part will contribute the dominant series to the asymptotic expansion. (It will be seen later, however, that only branch points ζ_b lying in the ζ-plane to the <u>right</u> of the image curve Γ of C contribute anything to the expansion.)

There is one serious difficulty in the method, however. As alluded to above, the nature of the series for z in the neighborhood of ζ_b depends upon the order of the first non-vanishing derivative of f at z_b. If, as indicated by the notation in (1.1) and (1.2), f happens to depend upon another parameter, α, then the order of this "first non-vanishing derivative" at a given branch point may be different for different values of α. From the point of view of practical applications the most interesting case is that in which the order, ν, is the same for all values of α in a neighborhood of a particular value, say $\alpha = 0$, but becomes greater than ν when $\alpha = 0$; i.e.

$f^{(\nu)}(z,a) \neq 0$ for all $a \neq 0$, but $f^{(\nu)}(z,0) = 0$.

For the sake of simplicity and definiteness we shall treat, in the present paper, the case in which the branch point is of first order ($\nu=2$) when $a \neq 0$ and of second order ($\nu=3$) when $a = 0$. More complicated cases seldom arise in practice and, in any event, it seems unlikely that branch points of higher order could cause any real theoretical difficulty, for the idea of the method to be presented seems capable of straightforward generalization.

Thus, let $f(z,a)$ be a function analytic in z in a neighborhood of the origin, $z = 0$. Let $z_{\pm}(a)$ be the two (distinct) zeros of $f'(z,a)$ closest to the origin. (Here and henceforth primes denote differentiation with respect to the variable other than a and we assume that for $a \neq 0$ there are exactly two such zeros); i.e. let

$$f'[z_{\pm}(a),a] = 0 \tag{1.3}$$

where $z_+(a) \neq z_-(a)$.

Assume, however, that

$$z_+(0) = z_-(0) = z_0 \text{ (say)}, \tag{1.4}$$

so that for $a \neq 0$,

$$f''[z_{\pm}(a),a] \neq 0 , \tag{1.5}$$

but, at $a = 0$,

$$f''[z_{\pm}(0),0] = 0. \tag{1.6}$$

We require, though, that

$$f'''[z_{\pm}(\alpha),\alpha] \neq 0 \qquad \text{for all } |\alpha| < A \tag{1.7}$$

(<u>including</u> $\alpha = 0$).

In this case, apart from an exponential factor, the saddle point method yields an asymptotic expansion in powers of $n^{-1/2}$ for $\alpha \neq 0$, but one in powers of $n^{-1/3}$ if $\alpha = 0$. The coefficients in the expansion for $\alpha \neq 0$ are naturally functions of α and one might well ask what happens to them if we let $\alpha \to 0$. The answer is not too surprising in view of the fact that the series for $\alpha = 0$ proceeds in different powers of n from that for $\alpha \neq 0$; the answer is that they blow up. In fact, the three different expansions, one for $\alpha < 0$, one for $\alpha = 0$, and one for $\alpha > 0$ in general have to be computed separately.

Now it is well known that one of the principal uses of asymptotic series is for the purpose of numerical computation and it might be expected that the existence of three different asymptotic series in n for three different ranges of the parameter α would lead to poor numerical results for values of α very close to $\alpha = 0$. This is indeed the case. As a result, numerous efforts have been made to procure asymptotic representations valid in the so-called "transition region" when α is close to zero. Most of these efforts, however, have been directed toward solving specific problems rather than the general one and, as a result, their derivation generally requires the use of

highly specialized techniques, techniques which can rarely be extended to many examples outside the particular case for which they were developed. Moreover, they seldom if ever yield full asymptotic expansions, but rather only one term or so, and their asymptotic character is often difficult to prove. It would therefore be highly desirable to have a uniform (in α) asymptotic expansion valid for all α in a neighborhood of $\alpha = 0$.

Only very recently, F. W. J. Olver [9]* has succeeded in obtaining such uniform expansions for those integrals of the type (1.1) which are solutions of a certain class of second order linear ordinary differential equations. Actually, he is interested in expansions for solutions for such equations and not for integrals of the type (1.1) so that in order to apply his results to an integral like (1.1) it would first be necessary to find the differential equation (if it exists - it may not) satisfied by (1.1). In some cases, such as Bessel functions, this can indeed be done. In others, it cannot. It would therefore still be quite desirable to obtain a uniform expansion for an integral such as (1.1).

In the present paper a method for obtaining such a uniform expansion will be presented. Differing considerably from Olver's method, it consists in replacing the usual change of variable, (1.2), by a slightly more complicated

* Numbers in square brackets refer to the bibliography at the end of the report.

one, namely

$$b + 3a^2w - w^3 = -f(z,\alpha) \tag{1.8}$$

where w is the new variable of integration and where (the reasons for the following choice of a and b are explained at length in §3.)

$$a^3 = \frac{1}{4}[f(z_-,\alpha) - f(z_+,\alpha)] \tag{1.9}$$

$$b = -\frac{1}{2}[f(z_+,\alpha) + f(z_-,\alpha)] \tag{1.10}$$

(cf. (1.3) - (1.7) for the definition of $z_\pm$).

Under the change of variable (1.8) the integral (1.1) goes over into

$$F(n,\alpha) = e^{nb} \int_K e^{n(3a^2w-w^3)}\, \phi(w,\alpha)dw\ , \tag{1.11}$$

where

$$\phi(w,\alpha) = \bar{\phi}[z(w,\alpha)]\, \frac{dz}{dw}\ , \tag{1.12}$$

$z(w,\alpha)$ being defined implicitly by (1.8), and where K is the image of C under (1.8).

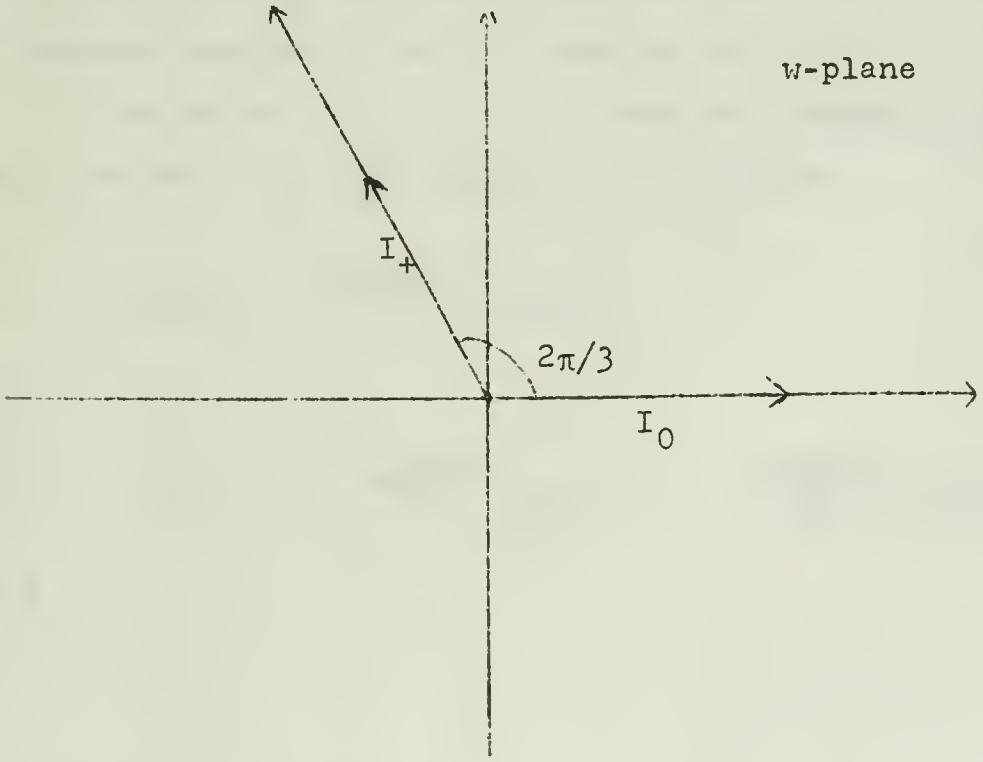
As a result of the choice of a and b the mapping (1.8) can be (and in §4 is) proved to be one to one in a neighborhood including both branch points $z_\pm$ provided α is sufficiently small. This means that z can be expanded into a convergent power series in w in this neighborhood.

Slightly more than this is needed. It is shown that ϕ can be expanded into a convergent series of the following type:

$$(1.13) \qquad \phi(w,\alpha) = \sum_{k=0}^{\infty} [c_k^{(1)}(a^2-w^2)^k + w\, c_k^{(2)}(a^2-w^2)^k] ,$$

the coefficients of which are not too hard to compute; i.e. a prescription for computing the coefficients can be given. In practice it might be rather difficult to compute more than the first few, but theoretically it can be done.

The path K of the integral in (1.11) is then deformed so as to pass through the region of validity of the series (1.13) and after a number of remainder estimates and some further path pushing it is shown that termwise integration of the series (1.13) over the path $L_+ = I_0 - I_+$ shown below



yields an asymptotic series of the form

$$(1.14) \qquad F(n,a) \sim e^{nb} \sum_{k=0}^{\infty} C_k(n,a)$$

$$= e^{nb} \sum_{k=0}^{\infty} [c_k^{(1)}(a)A_k(n,a) + c_k^{(2)}(a)B_k(n,a)]$$

where the $c_k^{(i)}$ $(i = 1,2)$ are the coefficients appearing in the series (1.13) and where

$$(1.15) \qquad A_k(n,a) = \int_{L_+} (a^2-w^2)^k \, e^{n(3a^2w-w^3)} \, dw$$

and

$$(1.16) \qquad B_k(n,a) = \int_{L_+} w(a^2-w^2)^k \, e^{n(3a^2w-w^3)} \, dw \, .$$

Note that A_k and B_k are really functions of n and α since, as remarked previously, a is a function of α.

If we define $A_{-1} = B_{-1} = 0$, then the integrals A_k and B_k satisfy recursion relations of the form

$$(1.17) \qquad A_k(n,a) = \frac{2(k-1)}{3n} B_{k-2}(n,a)$$

and

$$(1.18) \qquad B_k(n,a) = -\frac{(2k-1)}{3n} A_{k-1}(n,a) + \frac{2(k-1)a^2}{3n} A_{k-2}(n,a)$$

for $k \geq 1$.

By using (1.17) and (1.18) it will be shown later that the functions A_k and B_k and therewith C_k (cf. (1.14)) are of decreasing order of magnitude in n. For example,

$$(1.19)\qquad \begin{cases} C_k(n,\alpha) = O(n^{-\frac{k}{2}})\, C_0(n,\alpha) & \text{if } k \text{ is even} \\ C_k(n,\alpha) = O(n^{-\frac{k+1}{2}})\, C_0(n,\alpha) & \text{if } k \text{ is odd} \end{cases}$$

independently of α (see p. 52). This, plus an estimation of the remainder, (which is also carried out later), establishes the asymptotic character of the series (1.14).

The recurrence relations (1.17) and (1.18) also enable one to express all the A_k's and B_k's in terms of A_0 and B_0. Of course, all this would be of very little use if A_0 and B_0 were not themselves expressible in terms of some well known functions, functions sufficiently well tabulated to enable the formula (1.14) to be of use for numerical computations. Fortunately (and in fact this is one of the reasons for the choice of the transformation (1.8)) A_0 and B_0 are expressible in terms of well known functions. If

$$(1.20)\qquad Ai_+(z) = \frac{1}{2\pi i} \int_{L_+} e^{zt - \frac{1}{3}t^3}\, dt$$

where L_+ is the same path as shown on p.7; is one of the Airy integrals (see p. B-67 of Friedrichs' notes on Special Topics in Analysis [4] or Jeffreys and Jeffreys [7], p. 476), then

$$(1.21)\qquad A_0(n,a) = \frac{2\pi i}{(3n)^{1/3}}\, Ai_+[(3n)^{2/3}\, a^2]$$

and

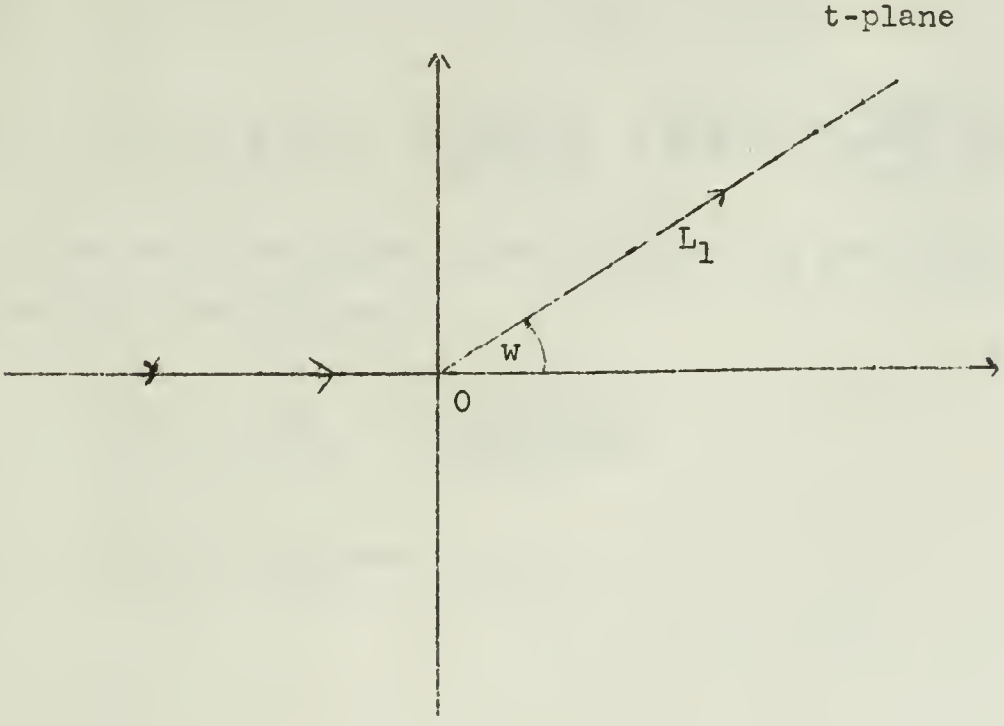
$$(1.22)\qquad B_0(n,a) = \frac{2\pi i}{(3n)^{2/3}}\, Ai'_+[(3n)^{2/3}\, a^2]$$

(prime denotes derivative with respect to the argument).

For the purposes of numerical computation it is somewhat more convenient to relate A_0 and B_0 to the following integral which is also a solution of Airy's differential equation:

$$(1.23)\qquad h_1(z) = \frac{k}{i\pi} \int_{L_1} e^{zt + \frac{1}{3}t^3}\, dt$$

(k here is a numerical constant and has no relation to any indices appearing in preceding formulae) where L_1 is the path shown below



and $\frac{\pi}{6} < w < \frac{\pi}{2}$ (again this w has no relation to our w). The integral $h_1(z)$ together with its derivative, $h_1'(z)$, is tabulated for complex values of the argument z in the Harvard Computational Laboratory's "Hankel Functions of Order $1/3$" [5]. A_0 and B_0 are expressed in terms of h_1 and h_1' by the following formulas:

$$(1.24) \qquad A_0(n,a) = \frac{\pi e^{\frac{i\pi}{6}}}{k(3n)^{1/3}} h_1[e^{-\frac{i\pi}{3}} (3n)^{\frac{2}{3}} a^2] ;$$

$$(1.25) \qquad B_0(n,a) = \frac{\pi e^{-\frac{i\pi}{6}}}{k(3n)^{2/3}} h_1'[e^{-\frac{i\pi}{3}} (3n)^{\frac{2}{3}} a^2].$$

Using all these formulas, theoretically any desired number of terms in the series (1.14) can be written down, although in practice seldom more than the first term is needed. For it, only $c_0^{(1)}(a)$ is needed, and (see §6)

$$(1.26) \quad c_0^{(1)}(a) = \frac{1}{2} \bar{\Phi}(z_+) \sqrt{\frac{6a}{f_{zz}(z_+,a)}} + \frac{1}{2} \bar{\Phi}(z_-) \sqrt{\frac{-6a}{f_{zz}(z_-,a)}} .$$

If we let $a \to 0$, than $a \to 0$ as does f_{zz} and $c_0^{(1)}(a)$ goes over continuously into

$$(1.27) \quad c_0^{(1)}(0) = \bar{\Phi}(z_0) \cdot \sqrt[3]{\frac{6}{f_{zzz}(z_0,0)}} .$$

Thus, by (1.14), (1.24) and (1.26),

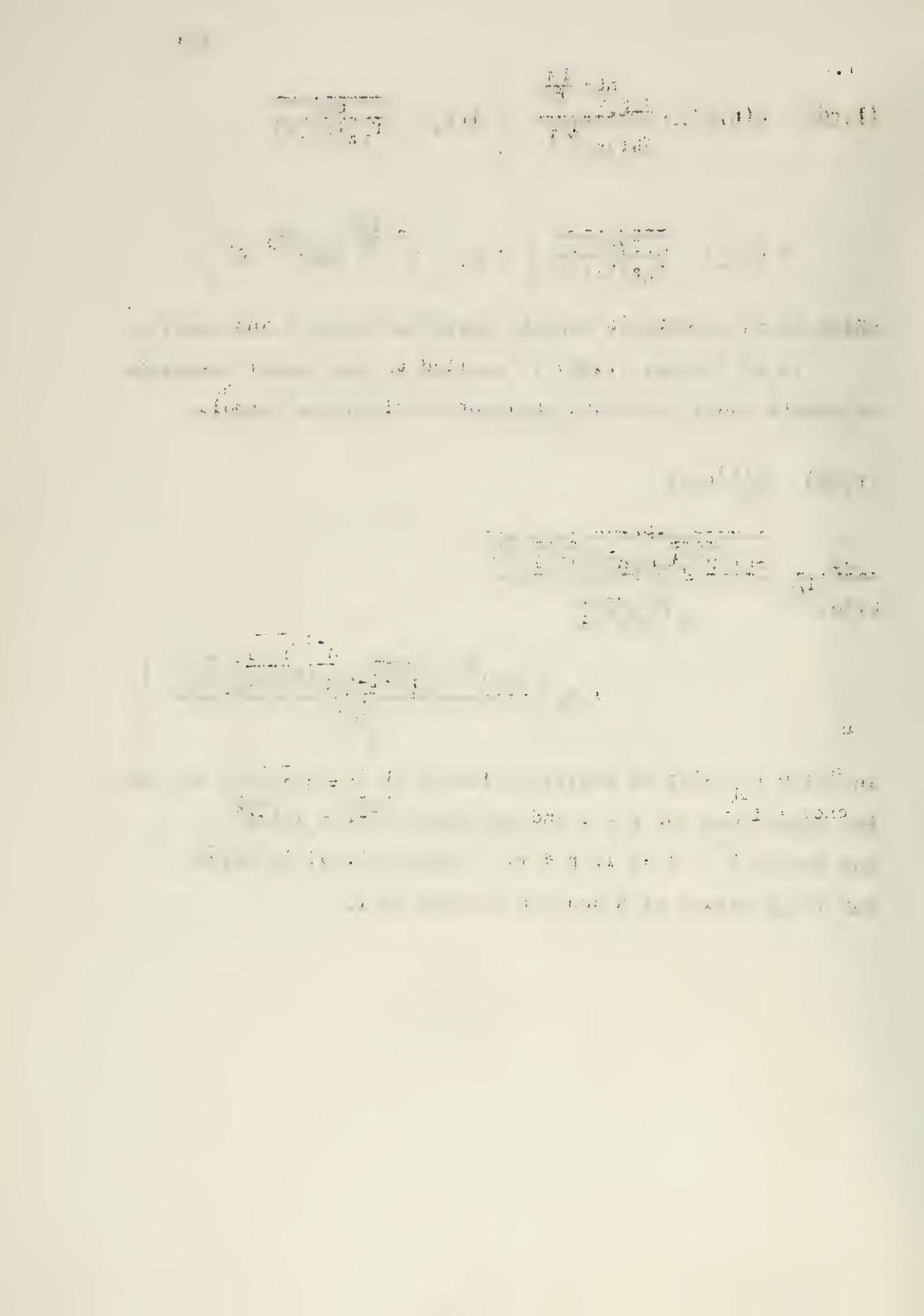
$$(1.28)\quad F(n,a) \sim \frac{\pi\, e^{nb+\frac{i\pi}{6}}}{2k(3n)^{1/3}} \left[\Phi(z_+) \sqrt{\frac{6a}{f_{zz}(z_+,a)}} + \Phi(z_-) \sqrt{\frac{-6a}{f_{zz}(z_-,a)}} \right] \cdot h_1 \left[e^{-\frac{i\pi}{3}} (3n)^{2/3}\, a^2 \right],$$

which is an asymptotic formula valid for large n and small a.

In s8 formula (1.28) is applied to the Hankel functions of nearly equal order and argument to yield the formula

$$(1.29)\quad H_n^{(1)}(nr) \sim \frac{-i}{k(3n)^{1/3}} \sqrt{\frac{-6(\sqrt{r^2-1}-\tan^{-1}\sqrt{r^2-1})^{1/3}}{2^{1/3}\sqrt{r^2-1}}} \cdot h_1 \left[\frac{(3n)^{2/3}(\sqrt{r^2-1}-\tan^{-1}\sqrt{r^2-1})^{2/3}}{2^{2/3}} \right]$$

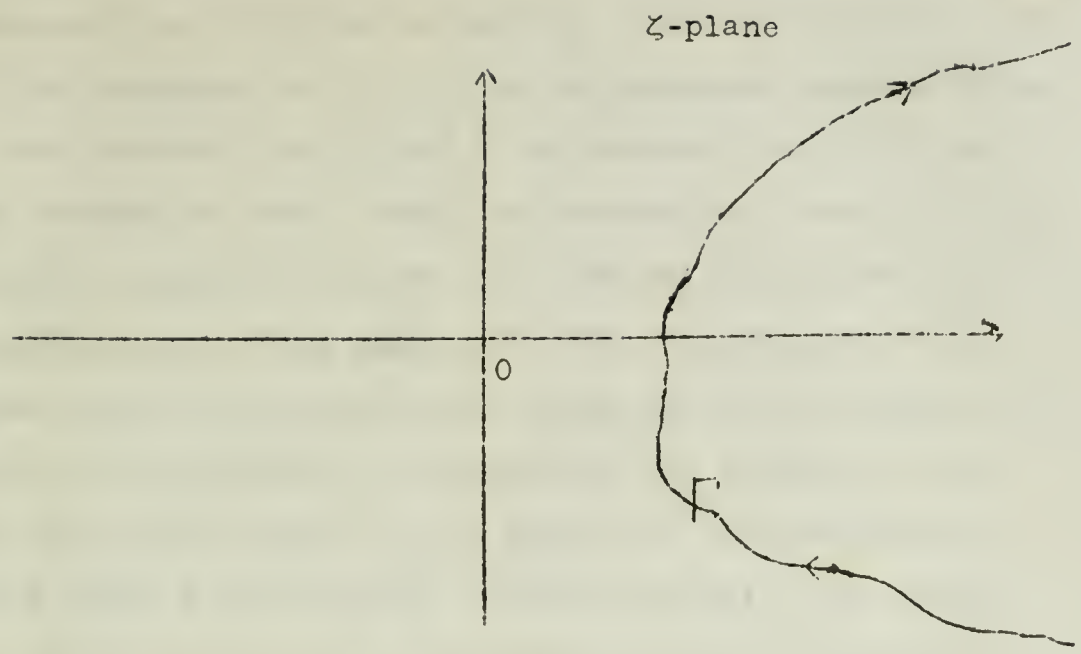
in which (as will be explained later) it is necessary to take the above form for $r > 1$ and to write $\sqrt{r^2-1} = i\sqrt{1-r^2}$ and choose $i^{1/3} = -i$ if $r < 1$. Formula (1.29) is valid for large values of n and for r close to 1.



§2. The Saddle Point Method

As mentioned in the introduction, there is a well known standard way for obtaining an asymptotic expansion for an integral of the type (1.1). This method, which was originated by Debye [3] in 1909, yields, apart from an exponential factor, an asymptotic series for F in fractional powers of n. As originally conceived by Debye and later elaborated upon by others the method possessed labyrinthal arguments about the nature of so-called "paths of steepest descent", paths passing through the cols of Re f and on which Re f decreases most rapidly upon leaving the cols. In addition to the difficulties in understanding the ideas of the method, there are formidable practical difficulties in computing the coefficients in the expansion. To date no one has succeeded in obtaining any essential simplification of the computational difficulties. However a considerable simplification in the presentation of the ideas of the method has been achieved recently by B. L. van der Waerden [10] who describes the saddle point method (as it is often called) as follows:

In (1.1) introduce the new complex variable $\zeta = f(z,a)$ as in (1.2). Then C goes over into some curve, Γ, in the w-plane which, because of the assumption that Re f > 0 far out on C, goes to infinity in the right half of the ζ-plane:



The integral (1.1) goes over into

$$(2.1) \qquad F(n,a) = \int_{\Gamma} e^{-n\zeta}\, \phi[z(\zeta,a)]\, \frac{dz}{d\zeta}\, d\zeta$$

or

$$(2.2) \qquad F(n,a) = \int_{\Gamma} e^{-n\zeta}\, \psi(\zeta,a)\, d\zeta\ ,$$

where

$$(2.3) \qquad \psi(\zeta,a) = \phi[z(\zeta,a)]\, \frac{dz}{d\zeta}\ .$$

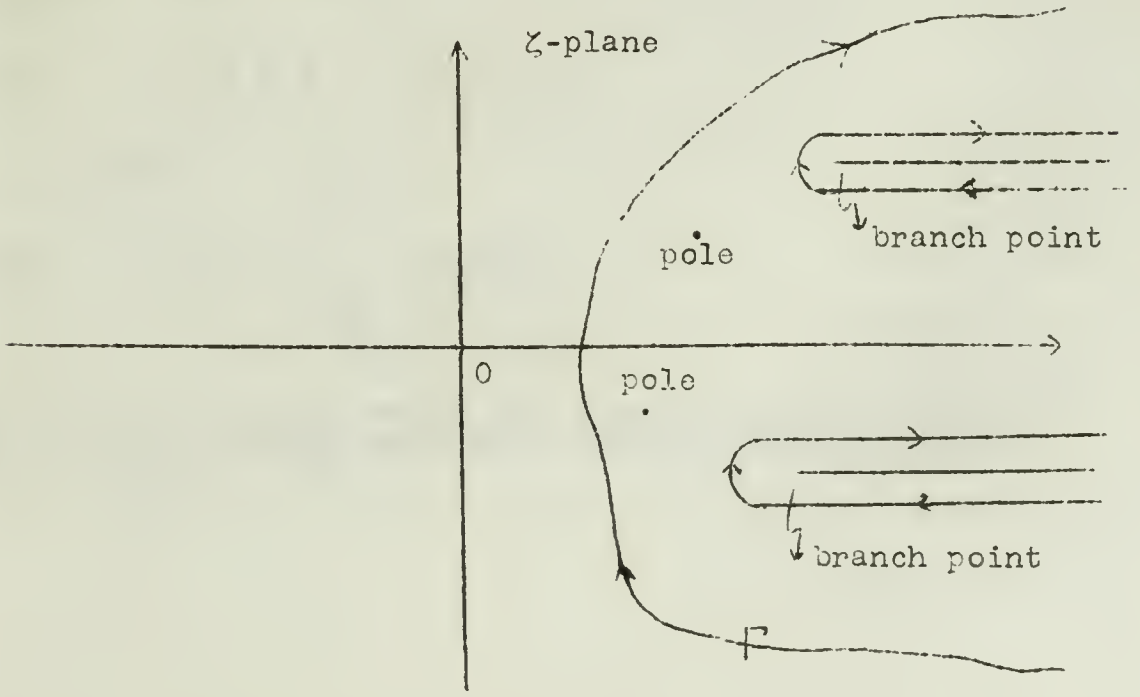
If the relatively mild assumption that

$$(2.4) \qquad |\psi(\zeta,a)| < Me^{k|\zeta|} \qquad (M \text{ and } k \text{ constants})$$

for Re ζ sufficiently large is made, then the path of integration, Γ, can be deformed in such a way as to yield the desired asymptotic series for F. It is well known that the values of complex integrals are determined by the

singularities of their integrands. Clearly the only part of the integrand of (2.2) that can have any singularities is the function $\psi(z,a)$ and it is obvious from (2.3) and the assumption that f and $\bar{\phi}$ are meromorphic that the only possible singularities of ψ are the poles of $\bar{\phi}$ and the branch points of the mapping (1.2). Furthermore, only those poles and branch points which lie to the right of Γ will be of importance in evaluating the integral (2.2). For each branch point to the right of Γ we introduce in the ζ-plane a cut parallel to the real axis. We assume for simplicity that no such branch cut meets the curve Γ, for otherwise we could always modify the cuts in order to achieve this.

Once this has been done it is then clear by Cauchy's theorem that the integral (2.2) is equivalent to a sum of integrals over the banks of the branch cuts plus a sum of residues from the poles.



If ζ_p is a typical pole, say for simplicity of order one, then it contributes a term of the form

$$(2.5) \qquad F_p(n,a) = e^{-n\zeta_p}\, \psi_p(\zeta_p,a)$$

(p here is merely a subscript; no differentiation is implied) to the asymptotic expansion of F.

On the other hand, a branch point, ζ_b, contributes a term of the form

$$(2.6) \qquad F_b(n,a) = \int_{L_b} e^{-n\zeta}\, \psi(\zeta,a)d\zeta \ ,$$

where L_b is a loop surrounding the branch cut from ζ_b to infinity. Now $f(z,a)$ is by hypothesis analytic in z. Again for simplicity, assume that f is analytic at the pre-image, z_b, of the branch point ζ_b. Then

$$(2.7) \qquad \zeta = f(z,a) = c_0(a) + c_\nu(a)(z-z_b)^\nu + \ldots$$

where

$$(2.8) \qquad c_\nu(a) \not\equiv 0 \quad \text{and} \quad \nu \geq 2.$$

Put[*]

$$(2.9) \qquad \alpha(z,a) = \int_{z_b}^{z} \Phi(\zeta)d\zeta$$

* This idea, which avoids the necessity of inverting the series for ψ and $\frac{dz}{d\zeta}$ separately, was suggested by Prof. A. S. Peters.

Then

$$(2.10) \qquad \omega_\zeta(z,a) = \bar{\phi}[z(\zeta,a)]\,\frac{dz}{d\zeta} = \psi(\zeta,a)\ .$$

Now by the Lagrange-Bürmann formula [6], the series (2.7) can be inverted and we can expand $\omega(z,a)$ in a series in $(\zeta-\zeta_b)^{\frac{1}{\nu}}$:

$$(2.11) \qquad \omega(z,a) = \sum_{\mu=0}^{\infty} \gamma_\mu(a)(\zeta-\zeta_b)^{\frac{\mu}{\nu}}\ ,$$

whence

$$(2.12) \qquad \psi(z,a) = \sum_{\mu=0}^{\infty} \gamma_\mu(a)(\zeta-\zeta_b)^{\frac{\mu}{\nu}}\ .$$

The series (2.12) will converge absolutely and uniformly in some neighborhood, $|\zeta-\zeta_b| < B$, of ζ_b. In such a neighborhood we can insert it into (2.6) and integrate term by term, obtaining

$$(2.13) \qquad F_b'(n,a) = \sum_{\mu=0}^{\infty} \gamma_\mu(a) \int_{L_b'} e^{-n\zeta}(\zeta-\zeta_b)^{\frac{\mu}{\nu}}\,d\zeta\ ,$$

where L_b' is that part of the path L which lies in the neighborhood in question.

Next, in order to be able to evaluate explicitly the integrals occurring in (2.13) we extend the path L_b' to infinity again (i.e. replace it by L_b). Such an extension is clearly permissible in virtue of the presence of the exponential $e^{-n\zeta}$ in the integrand. Of course, the resulting

series,

$$(2.14)\quad S_b(n,a) = \sum_{\mu=0}^{\infty} \gamma_\mu(a) \int_{L_b} e^{-n\zeta}(\zeta-\zeta_b)^{\frac{\mu}{\nu}}\, d\zeta$$

is no longer equal to F_b. It can be shown, however, and indeed we shall show it later on in a slightly different context, that the series S_b is an asymptotic expansion for F_b as $n \to \infty$. Moreover, the terms of S_b are easy to compute, for on the part of L_b which runs along the upper bank of the branch cut we can use the parameter

$$(2.15)\quad (\zeta-\zeta_b)^{\frac{1}{\nu}} = \tau^{\frac{1}{\nu}}, \quad 0 \leqq \tau < \infty,$$

whereas on the lower bank we can take

$$(2.16)\quad (\zeta-\zeta_b)^{\frac{1}{\nu}} = e^{\frac{2i\pi}{\nu}} \tau^{\frac{1}{\nu}}, \quad 0 \leqq \tau < \infty.$$

The series (2.14) then becomes

$$(2.17)\quad S_b(n,a) = \sum_{\mu=0}^{\infty} \gamma_\mu(a)(1 - e^{\frac{2i\pi\mu}{\nu}}) \int_0^{\infty} e^{-n(\zeta_b+\tau)} \tau^{\frac{\mu}{\nu}}\, d\tau$$

$$= e^{-n\zeta_b} \sum_{\mu=0}^{\infty} (1 - e^{\frac{2i\pi\mu}{\nu}})\, \gamma_\mu(a) \int_0^{\infty} e^{-n\tau} \tau^{\frac{\mu}{\nu}}\, d\tau$$

$$= e^{-n\zeta_b} \sum_{\mu=0}^{\infty} (1 - e^{\frac{2i\pi\mu}{\nu}})\, \gamma_\mu(a) \int_0^{\infty} e^{-n\tau} \frac{(n\tau)^{\frac{\mu}{\nu}} - 1}{n^{\frac{\mu}{\nu}}}\, d(n\tau)$$

$$= e^{-n\zeta_b} \sum_{\mu=0}^{\infty} (1 - e^{\frac{2\pi i\mu}{\nu}}) \frac{\gamma_\mu(a)\Gamma(\frac{\mu}{\nu})}{n^{\frac{\mu}{\nu}}} \quad .$$

Thus,

$$(2.18) \quad S_b(n,a) = e^{-n\zeta_b} \sum_{\mu=0}^{\infty} (1 - e^{\frac{2\pi i\mu}{\nu}}) \frac{\gamma_\mu(a)\Gamma(\frac{\mu}{\nu})}{n^{\frac{\mu}{\nu}}} \quad .$$

It is clear from (2.18) that the branch point ζ_b with smallest real part (i.e. closest to Γ) will make the major contribution to the asymptotic expansion of F, for the other branch points will all contribute series which are multiplied by factors exponentially smaller than that of the dominant series. If ζ_s is the branch point with smallest real part, then

$$(2.19) \quad F(n,a) \sim e^{-n\zeta_s} \sum_{\mu=0}^{\infty} (1 - e^{\frac{2\pi i\mu}{\nu}}) \frac{\gamma_\mu(a)\Gamma(\frac{\mu}{\nu})}{n^{\frac{\mu}{\nu}}}$$

is an asymptotic series for $F(n,a)$. Of course, our above discussion by no means constitutes a rigorous proof of this fact. We have merely tried to indicate the salient features of the situation because below we shall have to give a detailed proof of the asymptotic character of a somewhat different series and the method of proof will be almost the exact analogue of the one given by van der Waerden for the case above.

§3. A Difficulty in the Saddle Point Method

Having completed our sketch of van der Waerden's exposition of the saddle point method, we come now to the problem which is the reason for the present paper. It will be noted that although we have assumed that the integral $F(n,a)$ given by (1.1) depends upon a parameter a as well as upon n, we have up to the present time made no ostensible allowance for this fact. There has, however, been one crucial point where the parameter a entered in a non-trivial way, namely in the assertion (2.8). Since $c_\nu(a)$ is continuous in a and $\not\equiv 0$, it is clear that there will be ranges of a wherein $c_\nu(a) \neq 0$. There may, however, be some value of a for which $c_\nu(a) = 0$. We assume that this value of a is isolated and without loss of generality may take the point to be the origin, $a = 0$, i.e. we suppose that

$$(3.1) \qquad c_\nu(0) = 0 \ ,$$

but

$$(3.2) \qquad c_\nu(a) \neq 0 \quad \text{for} \quad 0 < |a| < A \qquad (A > 0).$$

Then it is not hard to see that the coefficients, $\gamma_\mu(a)$, of the series (2.12) blow up as $a \longrightarrow 0$. In other words, the entire procedure breaks down. Actually, things are not quite that bad, since for any fixed value of a an asymptotic expansion can always be obtained in the form (2.19), whereas for $a = 0$ a similar expansion with different

coefficients and proceeding in powers of $n^{-\frac{\mu}{\nu'}}$ where ν' is the smallest integer > 1 such that $c_{\nu'}(0) \neq 0$ (such an integer surely exists) can be found. However, the expansions for $a \neq 0$ (there are in general <u>two</u> of them, one for $a > 0$ and a different one for $a < 0$) do not go over continuously into that for $a = 0$ as $a \to 0$ in them. In fact, as mentioned previously, they do not go over continuously into anything; they blow up.

As explained in the introduction, there is need for a uniform expansion. With a view toward arriving at such an expansion, let us analyse a little more closely the situation described by (3.1) and (3.2). From (2.7) we see that

$$(3.3) \qquad c_k(a) = \frac{1}{k!} \left.\frac{d^k \zeta}{dz^k}\right|_{z=z_b} = \frac{1}{k!} \frac{\partial^k f(z_b, a)}{\partial z^k} \quad .$$

Relations (3.1) and (3.2) therefore mean that

$$(3.4) \qquad \frac{\partial^\ell f(z_b, a)}{\partial z^\ell} = 0 \qquad \ell = 1, 2, \ldots \nu-1, \text{ for all } a,$$

while

$$(3.5) \qquad \frac{\partial^\nu f(z_b, a)}{\partial z^\nu} \neq 0 \qquad \text{if } a \neq 0.$$

For $a = 0$, however,

$$(3.6) \qquad \frac{\partial^{\nu} f(z_b, a)}{\partial z^{\nu}} = 0 \ .$$

In other words, z_b is a branch point of order ν for $a \neq 0$ but becomes a branch point of order $> \nu$ for $a = 0$, and it is precisely this phenomenon that causes the breakdown of the series expansion (2.12) and the resulting failure of the saddle point method.

As mentioned in the introduction, we shall treat only the case in which two branch points of second order coalesce to become a branch point of third order, i.e. in which assumptions (1.3) - (1.7) are satisfied. From them it follows that in the neighborhood of the origin

$$(3.7) \qquad f(z,a) = c_0(a) + c_1(a)z + c_2(a)z^2 + c_3(a)z^3 + \ldots$$

where $c_1(a)$, $c_2(a) \neq 0$ for $a \neq 0$ but $c_1(0) = c_2(0) = 0$, while $c_3(a) \neq 0$ for all a in $|a| < A$ (<u>including</u> $a = 0$). Thus, if we put $\zeta = f(z,a)$, then z is a power series in ζ for $a \neq 0$, but a power series in $\zeta^{1/3}$ if $a = 0$. What is clearly needed is a uniform expansion for z in powers of a variable, say w, an expansion which retains its validity even for $a = 0$. Obviously we cannot merely set $w = \zeta$, but it is plausible that setting $\zeta =$ some function of w might do the trick. What we want is a function $\zeta = \zeta(w)$ such that the equation

$$(3.8) \qquad \zeta(w) = f(z,a)$$

has a power series solution

$$w = a_0(\alpha) + a_1(\alpha)z + a_2(\alpha)z^2 + \ldots \tag{3.9}$$

where

$$a_1(\alpha) \neq 0 \tag{3.10}$$

for all α in a neighborhood, $|\alpha| < A_0$, of $\alpha = 0$ (<u>including</u> $\alpha = 0$). Well, if $w = w(z)$ has a power series expansion such as (3.9), then

$$a_1(\alpha) = \left.\frac{dw}{dz}\right|_{z=0} . \tag{3.11}$$

From (3.8) we have

$$\zeta'(w)dw = f_z(z,\alpha)dz , \tag{3.12}$$

whence

$$a_1(\alpha) = \left.\frac{f_z(z,\alpha)}{\zeta'(w)}\right|_{z=0} . \tag{3.13}$$

Since f_z vanishes for $z = z_{\pm}(\alpha)$ and $z_{\pm}(\alpha) \longrightarrow 0$ as $\alpha \longrightarrow 0$, $a_1(\alpha)$ would seem to $\longrightarrow 0$ as $\alpha \longrightarrow 0$. There is, however, at least one way in which this undesired occurrence can be nullified. We can pick the function $\zeta(w)$ such that $\zeta'(w) = 0$ when $w = w_{\pm}$, where $w_{\pm}$ are the values of w which correspond to $z = z_{\pm}$ through equation (3.8), i.e. $\zeta(w_{\pm}) = f(z_{\pm},\alpha)$. This means that the function $\zeta'(w)$ must have zeros at at least two prescribed points; hence $\zeta(w)$ must be at least a cubic. Since two values of its derivative are prescribed,

we pick a cubic depending on two parameters, namely the following:

$$(3.14) \qquad g(w,a,b) \equiv b + 3a^2w - w^3 \ .$$

We therefore set

$$(3.15) \qquad g(w,a,b) = b + 3a^2w - w^3 = f(z,a) \ .$$

From (3.15) we have

$$(3.16) \qquad \frac{dw}{dz} = \frac{f_z(z,a)}{g_w(w,a,b)} = \frac{f_z(z,a)}{3(a^2-w^2)}$$

and therefore, since

$$(3.17) \qquad g_w = 0 \iff w = \pm a \ ,$$

we should pick a and b such that

$$(3.18) \qquad g(\pm a,a,b) = b \pm 2a^3 = f(z_{\pm},a)$$

since then f_z and g_w will vanish simultaneously.

Thus,

$$(3.19) \qquad b = \tfrac{1}{2}[f(z_+,a) + f(z_-,a)]$$

and

$$(3.20) \qquad a^3 = \tfrac{1}{4}[f(z_+,a) - f(z_-,a)] \ .$$

Equation (3.15) with a and b given by (3.19) and (3.20) defines w as an implicit function, $w = w(z,a)$. It will be proved that this function can be expanded into a convergent

power series in the neighborhood of the origin, a series which will be valid for all a in a neighborhood of $a = 0$, including $a = 0$, and which is such that the coefficient of z is different from zero for all such a. This last fact means, of course, that the inverse function $z = z(w,a)$ is also a power series in w good for all a in a neighborhood of $a = 0$.

We shall give a precise formulation and rigorous proof of this key theorem. For this purpose, it is more convenient to recast the formulation of the theorem a little bit. First of all we transform the origins of the z and w planes to one of the zeros of f_z and g_w, say z_+ and a, i.e. we introduce the new variables

$$x = z - z_+ \tag{3.21}$$

and

$$y = w - a \tag{3.22}$$

and we call

$$f(z,a) \equiv f(x + z_+,a) \equiv F(x,a) \tag{3.23}$$

and

$$\xi = z_- - z_+ \ . \tag{3.24}$$

Then, by (1.3) - (1.7), we have

$$\begin{aligned} F'(0,a) &= f'(z_+,a) = 0 \\ F'(\xi,a) &= f'(z_-,a) = 0 \end{aligned} \tag{3.25}$$

$$
\begin{aligned}
&F''(0,\alpha) = f''(z_+,\alpha) \neq 0 \quad \text{if } \alpha \neq 0 \\
&F''(\xi,\alpha) = f''(z_-,\alpha) \neq 0 \quad \text{if } \alpha \neq 0 \\
&F''(0,0) = f''(0,0) = 0
\end{aligned} \tag{3.26}
$$

$$
\begin{aligned}
&F'''(0,\alpha) = f'''(z_+,\alpha) \neq 0 , \quad (\text{all } \alpha) \\
&F'''(\xi,\alpha) = f'''(z_-,\alpha) \neq 0 \quad (\text{all } \alpha).
\end{aligned} \tag{3.27}
$$

Hence we can write

$$f(z,\alpha) = f(z_+,\alpha) + \tfrac{1}{2}(z-z_+)^2 f''(z_+,\alpha) + \tfrac{1}{6}(z-z_+)^3 f'''(z_+,\alpha) + \ldots \tag{3.28}$$

or

$$F(x,\alpha) = F(0,\alpha) + \tfrac{1}{2}x^2 F''(0,\alpha) + \tfrac{1}{6}x^3 F'''(0,\alpha) + \ldots , \tag{3.29}$$

where $F''(0,\alpha) \longrightarrow 0$ as $\alpha \longrightarrow 0$, but $F'''(0,\alpha)$ does not. There is no loss in generality in assuming that $\frac{1}{2}F''(0,\alpha) = \alpha$, for this merely amounts to replacing α by the new parameter $\frac{1}{2}F''(0,\alpha)$, a replacement which in no way affects any of the important features of the problem. Then calling

$$F(0,\alpha) = \beta, \tag{3.30}$$

we can write (3.29) in the form

$$F(x,\alpha) = \beta + \alpha x^2 + \gamma x^3 \rho(x,\alpha) , \quad \gamma \neq 0 , \tag{3.31}$$

where $\rho(0,0) = 1$ and $\rho(x,\alpha)$ is analytic in x and continuous in α in a neighborhood of $x = 0$, $\alpha = 0$.

Correspondingly, the function $g(w,a,b)$, defined by (3.14) goes over under the change of variable (3.22) into

$$g(w,a,b) = (b + 2a^3) - 3a\,y^2 - y^3 ,$$

or, letting

$$b^* = b + 2a^3 , \tag{3.32}$$

we have

$$g(w,a,b) \equiv G(y,a,b^*) \equiv b^* - 3a\,y^2 - y^3 . \tag{3.33}$$

Here, a is the same as the a in g. By (3.20), (3.23), (3.24) and (3.30) this means that

$$a^3 = \tfrac{1}{4}[f(z_+,\alpha) - f(z_-,\alpha)] , \tag{3.34}$$

or

$$a^3 = \tfrac{1}{4}[F(0,\alpha) - F(\xi,\alpha)] , \tag{3.35}$$

or

$$a^3 = \tfrac{1}{4}[\beta - F(\xi,\alpha] , \tag{3.36}$$

while b^* is obtained by using (3.19), (3.20), (3.23) and (3.32), as

$$b^* = b + 2a^3 = f(z_+,\alpha) = F(0,\alpha) = \beta . \tag{3.37}$$

The relations (3.36) and (3.37) giving a and b^* in terms of β and ξ could also have been obtained directly by requiring that G_y and F_x vanish at corresponding points, for from (3.25) we have

$$F'(x,\alpha) = 0 \quad \text{[illegible]} \quad x = 0 \text{ or } x = \xi \tag{3.38}$$

(for x and α in a sufficiently small neighborhood of $x = 0$, $\alpha = 0$), while from (3.33),

$$(3.39) \qquad G'(y,a,b^*) \equiv -3y(2a + y) \ ;$$

hence

$$(3.40) \qquad G'(y,a,b^*) = 0 \iff y = 0 \text{ or } y = -2a.$$

a and b^* are then to be picked such that

$$(3.41) \qquad G(0,a,b^*) = F(0,\alpha)$$

and

$$(3.42) \qquad G(-2a,a,b^*) = F(\xi,\alpha) \ ,$$

i.e. such that

$$(3.43) \qquad b^* = F(0,\alpha) = \beta$$

and

$$(3.44) \qquad b^* - 4a^3 = F(\xi,\alpha) \ ,$$

which agree with (3.36) and (3.37).

§4. The Key Theorem

Using (3.23), (3.31) and (3.33), equation (3.15) takes the form

$$(4.1) \qquad b^* - 3ay^2 - y^3 = \beta + \alpha x^2 + \gamma x^3 \rho(x,a), \quad \gamma \neq 0 .$$

But $b^* = \beta$, (see (3.37)), and therefore

$$(4.2) \qquad -3ay^2 - y^3 = \alpha x^2 + \gamma x^3 \rho(x,a) , \quad \gamma \neq 0 ,$$

which is the same equation as would be obtained if β were zero. We can therefore assume $\beta = 0$ to begin with and in addition we can replace F by -F which clearly does not alter the truth or falsity of the theorem we wish to prove. The equation to be treated is thus reduced to

$$(4.3) \qquad 3ay^2 + y^3 = \alpha x^2 + \gamma x^3 \rho(x,a) , \quad \gamma \neq 0 ,$$

where, since $\beta = 0$ and F has been replaced by -F,

$$(4.4) \qquad a = \tfrac{1}{4}F(\xi,a) = \tfrac{1}{4}[a\xi^2 + \gamma\xi^3\rho(\xi,a)] \quad \text{(by (3.36))} .$$

Having dispensed with the preliminaries, we can now formulate the main

Theorem: Let

$$(4.5) \qquad F(x,a) \equiv \alpha x^2 + \gamma x^3 \rho(x,a) , \quad \gamma \neq 0 ,$$

where $\rho(x,a)$ is analytic in x and continuous in a in a neighborhood of $x = 0$, $a = 0$, and where

$$(4.6) \qquad \rho(0,0) = 1 .$$

Let

$$G(y,a) \equiv 3ay^2 + y^3 , \tag{4.7}$$

and let ξ be the root of $F'(x,a) = 0$ (looked at as an equation in x) closest to but distinct for $a \neq 0$ from the origin, $x = 0$. (If a is sufficiently small, such a root always exists and is unique by virtue of the form (4.5) assumed for $F(x,a)$.

Then if and only if a is chosen such that

$$a^3 = \frac{1}{4}F(\xi,a) , \tag{4.8}$$

the equation

$$G(y,a) = F(x,a) , \tag{4.9}$$

i.e.

$$3ay^2 + y^3 = \alpha x^2 + \gamma x^3 \rho(x,a) , \quad \gamma \neq 0 , \tag{4.10}$$

has a unique solution,

$$y = H(x,a) \tag{4.11}$$

such that $H(0,a) = 0$ for all a in a neighborhood of $a = 0$ and such that $H(x,a)$ is a power series in x,

$$y = H(x,a) = h_1(a)x + h_2(a)x^2 + h_3(a)x^3 + \dots , \tag{4.12}$$

good for x and a in a neighborhood of $x = 0$, $a = 0$, a power series in which

$$h_1(a) \neq 0 \tag{4.13}$$

for all a in a neighborhood of $a = 0$. Note that here, as elsewhere in this discussion, we employ the usual definition of neighborhood; i.e., $a = 0$ is <u>included</u> in its own neighborhood. If we wish to indicate that $a = 0$ does not belong to its neighborhood, we shall use the standard term "deleted neighborhood" of $a = 0$.

The inequality (4.13) means, of course, that the inverse function, $x = H^{-1}(y,a)$, is also analytic in y in a neighborhood of $y = 0$ and has a power series expansion about $y = 0$ with a non-vanishing coefficient of y for all sufficiently small a. Thus, (4.11) gives a one to one conformal mapping of a neighborhood of the origin in the complex x-plane onto a neighborhood of the origin in the complex y-plane.

<u>Proof of the Theorem</u>:

We have already seen that the conditions of the Theorem are necessary. The real problem is to prove their sufficiency. To do this, we start in the natural way; we use (4.5) in (4.8). This gives

$$(4.14) \qquad a^3 = \frac{1}{4}[a\xi^2 + \gamma\xi^3\rho(\xi,a)] \ .$$

On the other hand, from the definition of ξ (see p. 30) we have

$$(4.15) \qquad 2a + 3\gamma\xi\rho(\xi,a) + \gamma\xi^2\rho'(\xi,a) = 0 \ .$$

If the value of ξ in terms of α determined by (4.15) is substituted into (4.14), then a is determined as a function of α. It is somewhat more convenient, however, to use (4.15) to eliminate α from (4.14) and look upon the resulting equation together with equation (4.15) as parametric equations for a and α in terms of the parameter ξ. Note that $\alpha \longrightarrow 0$ entails $\xi \longrightarrow 0$.

Using the abbreviations

$$\rho(\xi,\alpha) = p \quad \text{and} \quad \rho'(\xi,\alpha) = p' \tag{4.16}$$

(we can ignore the dependence of ρ upon α because ρ does not tend to zero with α) we obtain from (4.15)

$$\alpha = -\tfrac{3}{2}\gamma\,\xi\,p - \tfrac{1}{2}\gamma\,\xi^2 p' \quad , \tag{4.17}$$

or

$$\alpha = -\tfrac{1}{2}\gamma\xi(3p + \xi p') \ . \tag{4.18}$$

Substituting (4.17) into (4.14) we get

$$a = -\tfrac{1}{2}\gamma^{1/3}\,\xi(p + \xi p')^{1/3} \ . \tag{4.19}$$

Putting (4.18) and (4.19) back into (4.10) we then obtain

$$-\tfrac{3}{2}\gamma^{1/3}\,\xi(p + \xi p')^{1/3}\,y^2 + y^3 = -\tfrac{1}{2}\gamma\xi(3p + \xi p')x^2 + \gamma x^3\rho(x,\alpha) \ . \tag{4.20}$$

In (4.20), put

$$(4.21) \qquad y = \gamma^{1/3}(p + \xi p')^{1/3}\, x + v(x,\alpha) .$$

Then

$$-\tfrac{3}{2}\gamma(p + \xi p')\xi x^2 - 3\gamma^{2/3}(p + \xi p')^{2/3}\xi x v$$

$$-\tfrac{3}{2}\gamma^{1/3}(p + \xi p')^{1/3}\,\xi v^2 + \gamma(p + \xi p')x^3$$

$$+ 3\gamma^{2/3}(p + \xi p')^{2/3}\, x^2 v + 3\gamma^{1/3}(p + \xi p')^{1/3}\, x v^2$$

$$+ v^3 = -\tfrac{1}{2}\gamma(3p + \xi p')\xi x^2 + \gamma x^3 \rho(x,\alpha) ,$$

or

$$(4.22) \qquad 3\gamma^{2/3}(p + \xi p')^{2/3}(x-\xi)xv + 3\gamma^{1/3}(p + \xi p')^{1/3}(x-\tfrac{1}{2}\xi)v^2$$

$$+ v^3 = \gamma\,\{[\rho(x,\alpha) - (p + \xi p')]x + \xi^2 p'\}\, x^2 .$$

But, by Taylor's Theorem,

$$(4.23) \qquad \rho(x,\alpha) = p + (x - \xi)p' + (x - \xi)^2 R(x,\xi) .$$

Therefore the coefficient of x^2 in (4.22) can be written in the form

$$\gamma\,\{[(x - \xi)p' + (x - \xi)^2 R(x,\xi) - \xi p']x + \xi^2 p'\}$$

$$= \gamma(x - \xi)^2[p' + xR(x,\xi)] ,$$

so that (4.22) becomes

$$(4.24)\quad 3\gamma^{2/3}(p+\xi p')^{2/3}(x-\xi)xv + 3\gamma^{1/3}(p+\xi p')^{1/3}(x - \tfrac{1}{2}\xi)v^2 + v^3 = \gamma(x-\xi)^2\,[p' + xR(x,\xi)]x^2 .$$

In (4.24) put

$$(4.25)\quad v = x(x-\xi)u .$$

Then (4.24) becomes

$$(4.26)\quad 3\gamma^{2/3}(p+\xi p')^{2/3}u + 3\gamma^{1/3}(p+\xi p')^{1/3}(x - \tfrac{1}{2}\xi)u^2 + x(x-\xi)u^3 = \gamma[p' + xR(x,\xi)] .$$

Since $p = \rho(\xi,\alpha) \longrightarrow 1$ as $\alpha \longrightarrow 0$, the coefficient of the lowest order term in u in (4.26) is bounded away from zero as $\alpha \longrightarrow 0$, which, by the remark at the top of p. 32 is therefore also true as $\xi \longrightarrow 0$. Since equation (4.26) is of the form $\Omega(u,x,\alpha) = 0$ where Ω is analytic in u and x and continuous in α and where $\Omega(0,0,\alpha) = 0$, this means that $\Omega_u(0,0,\alpha) \neq 0$ for all sufficiently small α. The solution $u = u(x,\alpha)$ to such an equation is given by (see [1])

$$(4.27)\quad u = u(x,\alpha) = \frac{1}{2\pi i}\int_{|u|=\varepsilon} \frac{u\,\frac{\partial\Omega(u,x,\alpha)}{\partial u}\,du}{\Omega(u,x,\alpha)} ,$$

where $|u|=\varepsilon$ is a circle sufficiently small to ensure that $\Omega(u,x,\alpha) \neq 0$ thereon (since $\Omega_u(0,0,\alpha) \neq 0$, $\Omega(u,x,\alpha) \neq 0$ in a neighborhood of $u = 0$ and such a circle can therefore

always be chosen.)

From (4.27) it follows that u is analytic in x and continuous in α. The coefficients, $u_m(\alpha)$, in the power series expansion for u:

$$(4.28) \quad u(x,\alpha) = \sum_{m=0}^{\infty} u_m(\alpha)x^m$$

are given according to Cauchy's Theorem by

$$u_m(\alpha) = \frac{m!}{2\pi i} \int_{|x|=\delta} \frac{u(x,\alpha)}{x^{m+1}}\, dx$$

and are therefore continuous in α because u is.

From (4.25) and (4.28) it follows that

$$(4.29) \quad v(x,\alpha) = \sum_{m=0}^{\infty} u_m(\alpha)x^{m+1}(x-\xi) ,$$

and therefore by (4.21) that

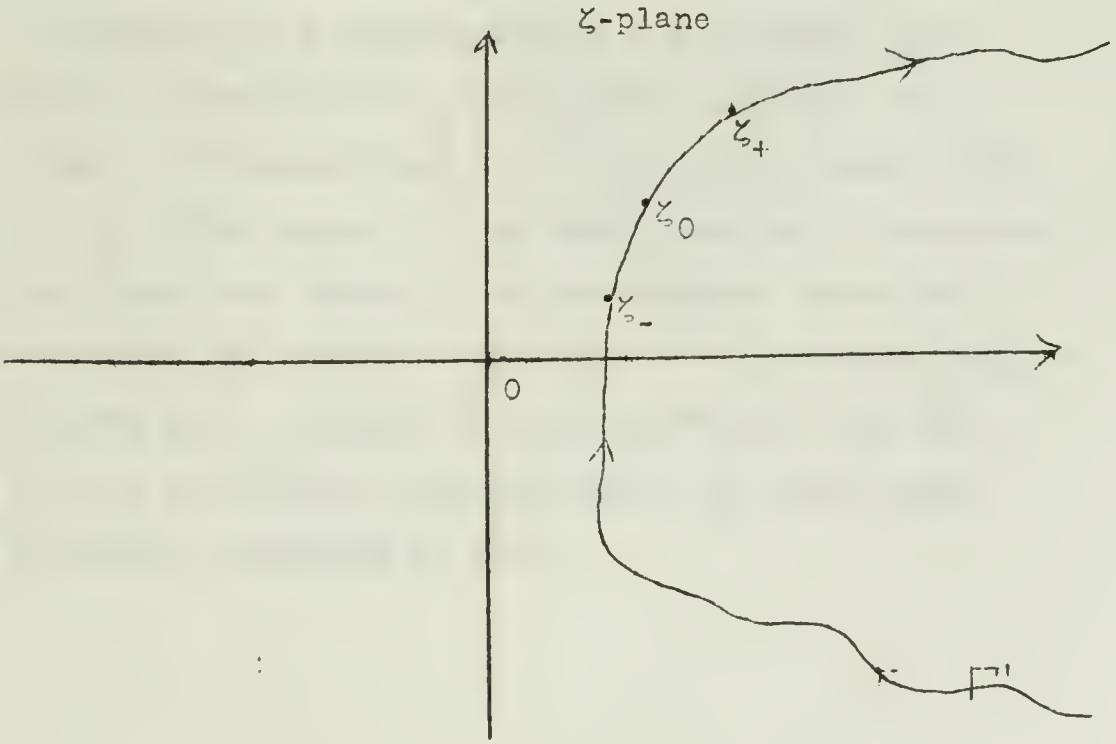
$$(4.30) \quad y = [\gamma^{1/3}(p + \xi p')^{1/3} - \xi\, u_0(\alpha)]x$$

$$+ x^2[u_0(\alpha) + \sum_{m=1}^{\infty} u_m(\alpha)x^{m-1}(x-\xi)] .$$

Since $\xi \to 0$ and $p \to 1$ as $\alpha \to 0$, it is clear that the coefficient of x in (4.29) is different from zero for all α sufficiently small and surely for $\alpha = 0$. Thus, if we take $H(x,\alpha)$ as the power series given by the right hand side of (4.29) we see that this $H(x,\alpha)$ satisfies all the desired conditions and the proof of the main theorem is therefore complete.

§5. Derivation and Proof of the Asymptotic Expansion

With the aid of the foregoing Theorem we are now in a position to extend the saddle point method to the case of the coalescence of two branch points. We start out exactly as van der Waerden does by making the change of variable (1.2) in the integral (1.1). As in §2, this gives an integral of the type (2.2). Next assume that the two branch points of the mapping (1.2) which lie closest to the curve Γ on the right depend upon α in such a way that they coalesce when $\alpha = 0$; call these two points $\zeta_{\pm}(\alpha)$ and their preimages $z_{\pm}(\alpha)$. Since $\zeta_{\pm}(\alpha)$ are the two branch points closest to Γ on the right, we can deform Γ into a path Γ' which passes through ζ_+ and by bending it a little if necessary, we can make Γ' pass through $\zeta_0=\zeta_+(0)=\zeta_-(0)$, the point of coalescence, too.



The deformation of Γ into Γ' is to be carried out in such a way that no branch points or poles of the integrand are crossed over in the process. As remarked above, the properties of $\zeta_{\pm}$ and ζ_0 ensure that this can always be done.

Finally, we assume that the $z_{\pm}(a)$ satisfy all the conditions listed on pp. 3-4. Then we know by the Theorem and the discussion preceding it on pp. 24-29 that if we perform the additional mapping

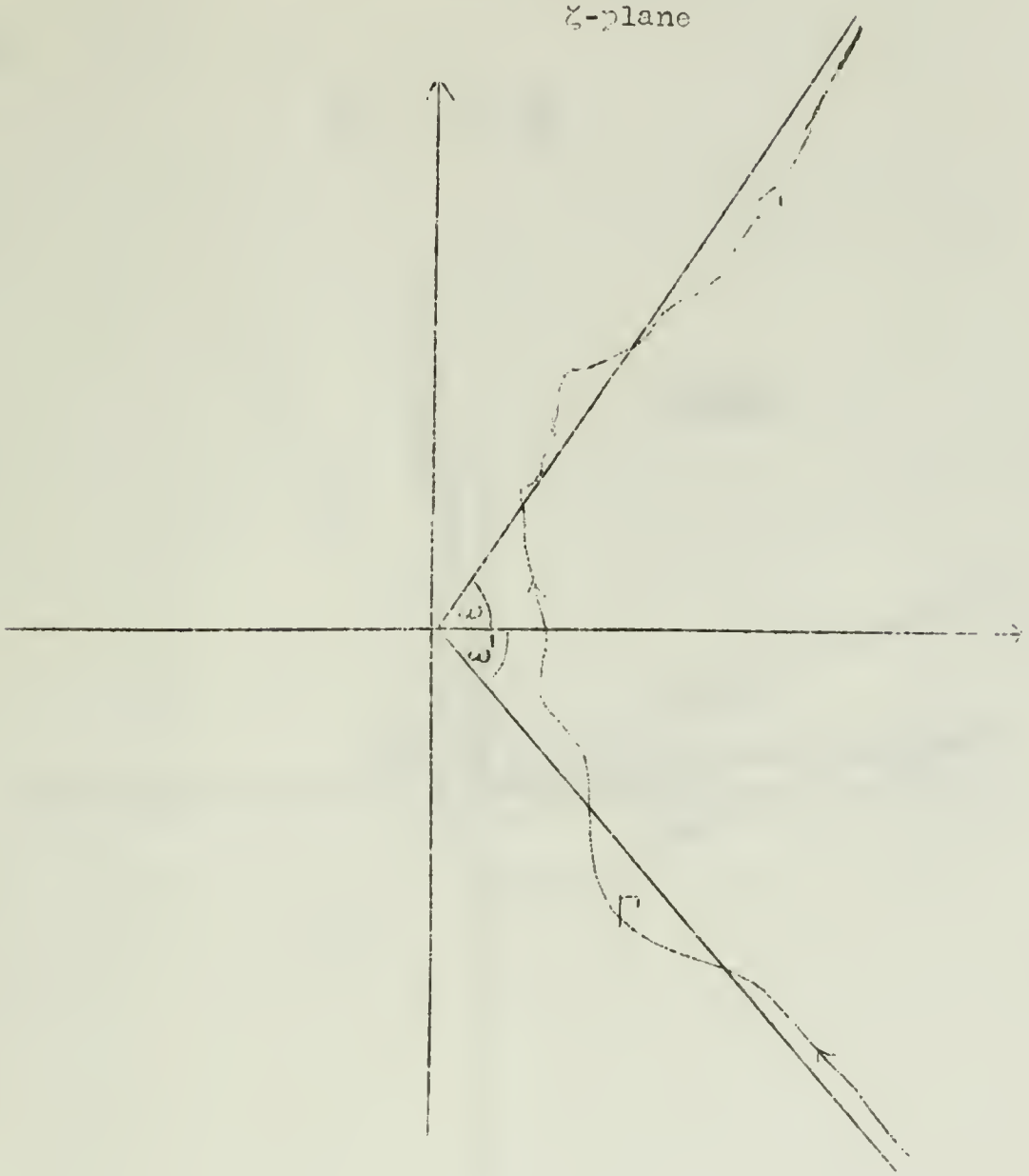
$$(5.1) \qquad g(w,a,b) = b + 3a^2w - w^3 = -\zeta = -f(z,a) ,$$

then $z_{\pm}$ are <u>not</u> branch points of the mapping from the z-plane to the w-plane. All other branch points of (1.2) remain branch points under (5.1), however, for $g_w = 0$ only at $w = \pm a$, and these points correspond to $z_{\pm}$. At all other points $z_\nu \neq z_{\pm}$ such that $f'(z_\nu,a) = 0$, $g_w \neq 0$, and such points therefore remain branch points.

Since $w = \pm a$ corresponds to $z = z_{\pm}$ under (5.1) and since by hypothesis $z_{\pm} \longrightarrow z_0$ and $a \longrightarrow 0$ as $a \longrightarrow 0$, cf. (1.4), it follows that $w = 0$ corresponds under (5.1) to $z = z_0$. The curve, Γ', in the ζ-plane will therefore go over into some curve, K, in the w-plane, which will pass through the points $w = \pm a$ and $w = 0$ and which will once again go to infinity in both directions. In fact, it is easy to determine the directions in which K goes to infinity. According to (5.1)

$w \sim \zeta^{1/3}$ for large ζ.

Now on Γ, $\zeta \sim |\zeta| e^{i\omega}$ in one direction (namely in the upper half-plane), where $0 < \omega < \frac{\pi}{2}$ and $\zeta \sim |\zeta| e^{(2\pi - \omega')i}$ in the other direction, where $0 < \omega' < \frac{\pi}{2}$.



Consequently, if $w \sim |w| e^{i\Omega}$ in one direction on K, and $w \sim |w| e^{i\Omega'}$ in the other direction on K, then

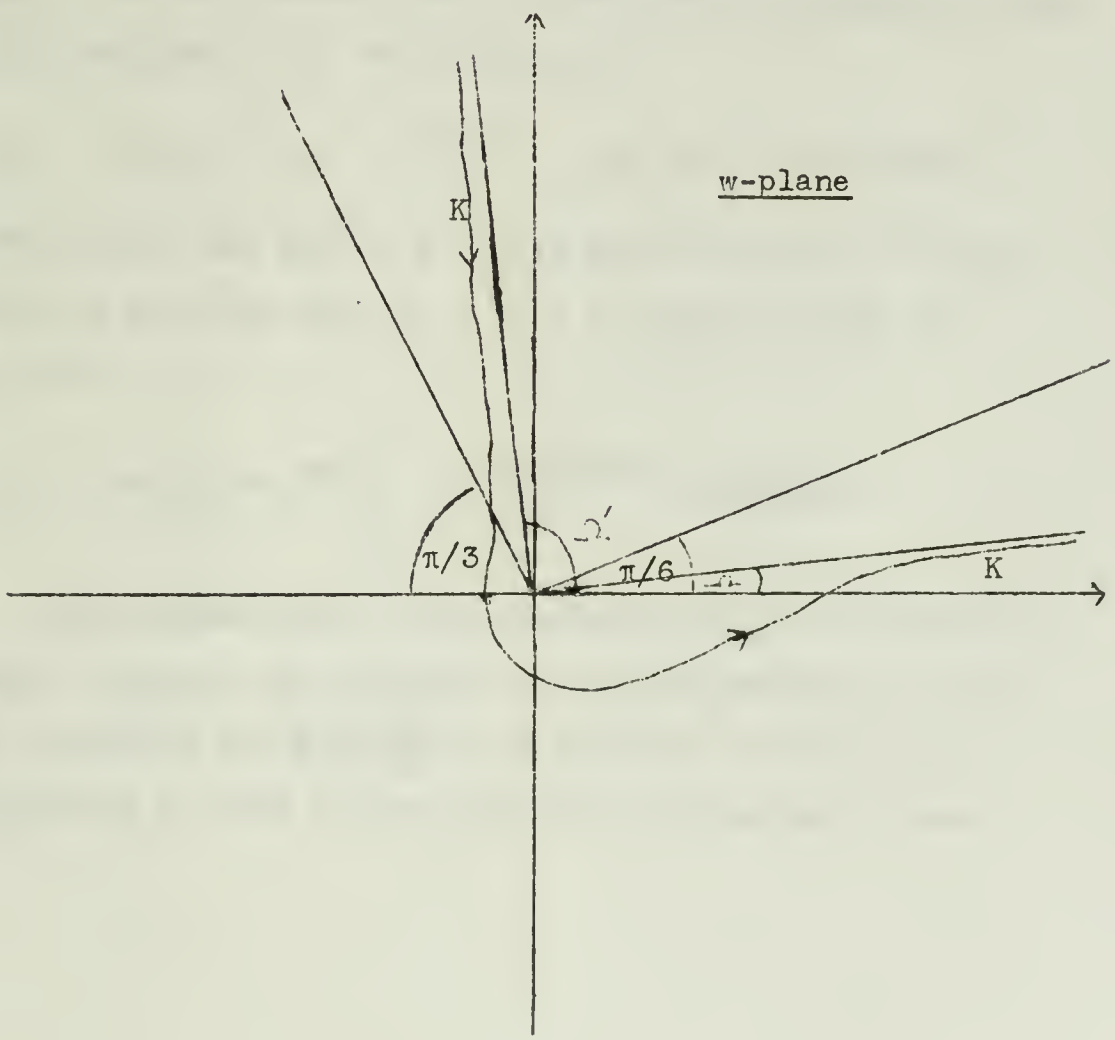
$$\Omega = \frac{\omega}{3} \; ; \; \Omega' = \frac{2\pi - \omega'}{3} \; ,$$

whence

$$0 < \Omega < \frac{\pi}{6}$$

and

$$\frac{\pi}{2} < \Omega' < \frac{2\pi}{3} \; .$$



Therefore K is a curve looking something like the one pictured on the preceding page. The integral (1.1), meanwhile, goes over into

$$(5.2) \qquad F(n,a) = e^{nb} \int_K e^{n(3a^2w-w^3)}\, \phi(w,a)dw \ ,$$

where

$$(5.3) \qquad \phi(w,a) = \underline{\Phi}[z(w,a)]\frac{dz}{dw} \ .$$

Again we need to impose a mild restriction such as (2.4). In the present case an even weaker assumption than (2.4) suffices; viz. we assume that

$$(5.4) \qquad |\phi(w,a)| < M' e^{\kappa|w|^3} \qquad (M' \text{ and } \kappa \text{ constants})$$

sufficiently far out on K. This being the case, we can deform K into the path $L_+ = I_0 - I_+$ shown on page 7, so that

$$(5.5) \qquad F(n,a) = e^{nb} \int_{L_+} e^{n(3a^2w-w^3)}\, \phi(w,a)dw \ .$$

Now assume that a (and therewith a) is sufficiently small to ensure the validity of the Main Theorem. Then z and therewith $\phi = \underline{\Phi}(z,a)\frac{dz}{dw}$ is an analytic function of w, possessing a power series expansion converging in some

circle $|w| < B'$ about the origin. If a is sufficiently small, then the point $w = 3a$ and therefore surely $w = a$ lies within this circle (note that our proof of the Theorem showsthat the radius of the circle cannot tend to zero as $a \longrightarrow 0$; this, in fact, is the important point of the Theorem). Hence ϕ is analytic in a neighborhood of $w = a$. So if we write

$$(5.6) \qquad \phi(w,a) = \phi_1(w^2,a) + w\,\phi_2(w^2,a) ,$$

(which can always be done), then $\phi_1(\eta,a)$ and $\phi_2(\eta,a)$ are analytic in a neighborhood of $\eta = a^2$ and consequently possess power series expansions of the form

$$(5.7) \qquad \phi_i(\eta,a) = \sum_{k=0}^{\infty} c_k^{(1)}(a)(a^2 - \eta)^k , \quad i = 1,2,$$

whence

$$(5.8) \qquad \phi_i(w^2,a) = \sum_{k=0}^{\infty} c_k^{(i)}(a)(a^2-w^2)^k , \quad i = 1,2.$$

As explained above, the series (5.8) converge in a neighborhood $|w| < B'$ of the origin including the points $w = \pm a$. We can therefore write

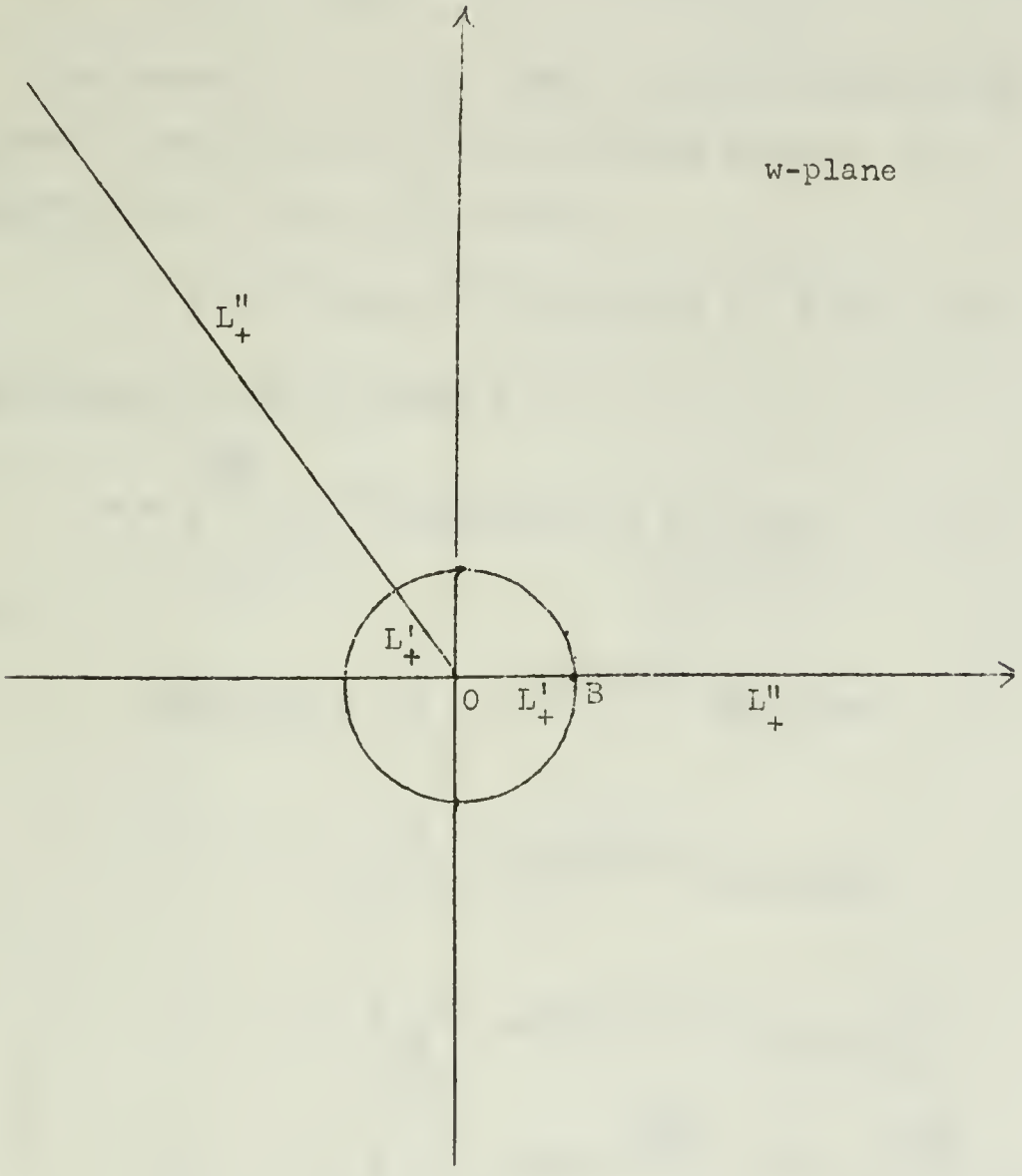
$$(5.9) \qquad \phi_i(w^2,a) = \sum_{k=0}^{K} c_k^{(i)}(a)(a^2-w^2)^k + r_i(w^2,a) , \quad (i = 1,2),$$

where

(5.10) $$r_i(w^2,a) = \Lambda(w,a)(a^2-w^2)^{K+1}$$

where $|\Lambda(w,a)| < \Lambda$ for $|w| < B'$.

Next take a circle $|w| = B$ of radius $|3a| < B < B'$ about the origin and denote by L_+' the portion of L_+ lying in this circle and by L_+'' the remainder of L_+. Then we can write



$$(5.11)\qquad F(n,a) = e^{nb}[F_1(n,a) + F_2(n,a)] ,$$

where

$$(5.12)\qquad F_1(n,a) = \int_{L_+'} e^{n(3a^2w-w^3)}\ \phi(w,a)dw$$

and

$$(5.13)\qquad F_2(n,a) = \int_{L_+''} e^{n(3a^2w-w^3)}\ \phi(w,a)dw \quad .$$

The behavior of F_2 for large n can be estimated immediately. On the part I_0'' of L_+'' running along I_0 (cf. diagrams on p. 7 and p. 42) we have

$$w = \tilde{\tau}, \text{ where } \tilde{\tau} \text{ is real and } 0 < B \leqq \tilde{\tau} < \infty .$$

On the part, I_+'', of L_+'' along I_+,

$$w = e^{\frac{2\pi i}{3}}\tilde{\tau}, \ \tilde{\tau} \text{ real and } 0 < B \leqq \tilde{\tau} < \infty .$$

Hence

$$\begin{aligned} |F_2(n,a)| &\leqq \left| \int_{I_0''} e^{n(3a^2w-w^3)}\ \phi(w,a)dw \right| \\ &+ \left| \int_{I_+''} e^{n(3a^2w-w^3)}\ \phi(w,a)dw \right| \\ &= \left| \int_B^\infty e^{n(3a^2\tilde{\tau}-\tilde{\tau}^3)}\ \phi(\tilde{\tau},a)d\tilde{\tau} \right| \\ &+ \left| \int_B^\infty e^{n(3a^2 e^{\frac{2\pi i}{3}}\tilde{\tau}-\tilde{\tau}^3)}\ \phi(e^{\frac{2\pi i}{3}}\tilde{\tau},a)d\tilde{\tau} \right| . \end{aligned}$$

But since $|3a| < B$,

(5.14) $\quad |3a^2\tau| < \frac{1}{3}B^2|\tau| \leq \frac{1}{3}|\tau|^3 \qquad$ for $|\tau| \geq B$.

Similarly, $|3a^2 e^{\frac{2\pi i}{3}}\tau| < \frac{1}{3}|\tau|^3$. Furthermore, by (5.4)

(5.4) $\quad |\phi(w,a)| < M' e^{K|w|^3}$

on the path. (τ is real and positive; so we can omit the absolute value signs). Therefore if $n > \frac{3K}{2}$,

(5.15) $\quad |F_2(n,a)| < 2M' \int_B^\infty e^{(K - \frac{2}{3}n)\tau^3} \, d\tau$.

If $B \geq 1$, then it follows from (5.15) that

$$|F_2(n,a)| < 2M' \int_B^\infty 3\tau^2 e^{(K - \frac{2n}{3})\tau^3} \, d\tau$$

$$= \frac{2M'}{\frac{2n}{3} - K} e^{-(\frac{2n}{3} - K)B^3} = \frac{2M' e^{KB^3}}{\frac{2n}{3} - K} e^{-\frac{2n}{3}B^3} .$$

Thus, if $n > \frac{3K}{2} + 1$,

$$|F_2(n,a)| < 2M' e^{KB^3} e^{-\frac{2n}{3}B^3} .$$

Calling

$$2M' e^{KB^3} = M^*$$

and

$$\frac{2B^3}{3} = q,$$

we have

$$|F_2(n,a)| < M^* e^{-qn}$$

for $n > \frac{3\kappa}{2} + 1$. In other words,

$$F_2(n,a) = O(e^{-qn}). \qquad (q > 0) \tag{5.16}$$

On the other hand, if $B < 1$, then we write (5.15) in the form

$$|F_2(n,a)| < 2M' \left[\int_B^1 e^{(\kappa - \frac{2}{3}n)\tau^3} \, d\tau + \int_1^\infty e^{(\kappa - \frac{2}{3}n)\tau^3} \, d\tau \right] .$$

The second integral here is $O(e^{-qn})$ by the argument just given above. As for the first, we have

$$\int_B^1 e^{(\kappa - \frac{2}{3}n)\tau^3} \, d\tau \leq e^{(\kappa - \frac{2}{3}n)B^3} \int_B^1 d\tau$$

$$= (1-B)e^{(\kappa - \frac{2}{3}n)B^3} = O(e^{-qn}) .$$

Thus, in any case, (5.16) holds.

The estimate (5.16) will enable us to show that the dominant contribution to the asymptotic expansion of F is made by F_1. In fact, the actual expansion is obtained from F_1 by inserting the series expansions, (5.9), into (5.6) and the result into (5.12). This gives

$$(5.17)\quad F_1(n,a) = \sum_{k=0}^{K} c_k^{(1)}(a) \int_{L_+'} (a^2-w^2)^k e^{n(3a^2w-w^3)}\, dw$$

$$+ \sum_{k=0}^{K} c_k^{(2)}(a) \int_{L_+'} (a^2-w^2)^k\, we^{n(3a^2w-w^3)}\, dw$$

$$+ \int_{L_+'} r_1(w^2,a)\, e^{n(3a^2w-w^3)}\, dw$$

$$+ \int_{L_+'} r_2(w^2,a)\, we^{n(3a^2w-w^3)}\, dw$$

$$= \sum_{k=0}^{K} c_k^{(1)}(a)\, \tilde{A}_k(n,a) + \sum_{k=0}^{K} c_k^{(2)}(a)\, \tilde{B}_k(n,a)$$

$$+ R_1(n,a) + R_2(n,a).$$

In order to transform the integrals $\tilde{A}_k$ and $\tilde{B}_k$ into integrals which can be easily evaluated, write

$$(5.18)\quad \tilde{A}_k(n,a) = \int_{L_+'} (a^2-w^2)^k\, e^{n(3a^2w-w^3)}\, dw$$

$$= \int_{L_+} (a^2-w^2)^k\, e^{n(3a^2w-w^3)}\, dw - \int_{L_+''} (a^2-w^2)^k e^{n(3a^2w-w^3)} dw.$$

Since

$$|a^2-w^2|^k < M'\, e^{K|w|^3},$$

the same argument as used on $F_2(n,a)$ (cf. pp. 43-45) applies to the second integral here and shows that it is $O(e^{-qn})$. Hence

$$(5.19)\quad \widetilde{A}_k(n,a) = \int_{L_+} (a^2-w^2)^k \, e^{n(3a^2w-w^3)} \, dw + O(e^{-qn}) \quad .$$

Similarly,

$$(5.20)\quad \widetilde{B}_k(n,a) = \int_{L_+} w(a^2-w^2)^k \, e^{n(3a^2w-w^3)} \, dw + O(e^{-qn}) \, .$$

So if we call

$$(5.21)\quad A_k(n,a) = \int_{L_+} (a^2-w^2)^k \, e^{n(3a^2w-w^3)} \, dw$$

and

$$(5.22)\quad B_k(n,a) = \int_{L_+} w(a^2-w^2)^k \, e^{n(3a^2w-w^3)} \, dw \, ,$$

we have

$$(5.23)\quad F_1(n,a) = \sum^{K} [c_k^{(1)}(a)A_k(n,a) + c_k^{(2)}(a)B_k(n,a)]$$

$$+ R_1(n,a) + R_2(n,a) + O(e^{-qn}) \, ,$$

or, with the further abbreviation

$$(5.24)\quad C_k(n,a) = c_k^{(1)}(a)A_k(n,a) + c_k^{(2)}B_k(n,a) \, ,$$

$$(5.25)\quad F_1(n,a) = \sum_{k=0}^{K} C_k(n,a) + R_1(n,a) + R_2(n,a) + O(e^{-qn}).$$

We wish to determine the order of the terms in (5.25) for large n. To do this we first integrate (5.21) by parts. If $k \geq 2$, we obtain

$$A_k(n,a) = -\frac{1}{3n} \int_{L_+} (k-1)(-2w)(a^2-w^2)^{k-2} e^{n(3a^2w-w^3)}\, dw.$$

Thus,

$$(5.26)\quad A_k(n,a) = \frac{2(k-1)}{3n} B_{k-2}(n,a)\ .$$

If $k = 1$, then $A_1(n,a) = 0$ because then the integrand is a perfect differential. Hence, if we agree to define $B_{-1} = 0$, (5.26) holds for $k \geq 1$.

Similarly, by integrating (5.22) by parts, we get

$$(5.27)\quad B_k(n,a) = -\frac{1}{3n} \int_{L_+} e^{n(3a^2w-w^3)} \frac{d}{dw}[w(a^2-w^2)^{k-1}]\, dw$$

$$= -\frac{1}{3n} \int_{L_+} e^{n(3a^2w-w^3)} (a^2-w^2)^{k-1}\, dw$$

$$+ \frac{2(k-1)}{3n} \int_{L_+} e^{n(3a^2w-w^3)}\, w^2 (a^2-w^2)^{k-2}\, dw \quad (\text{if } k \geq 2)$$

$$= -\frac{1}{3n} A_{k-1}(n,a) - \frac{2(k-1)}{3n} A_{k-1}(n,a) + \frac{2(k-1)a^2}{3n} A_{k-2}(n,a),$$

so that

$$(5.28)\quad B_k(n,a) = \frac{-(2k-1)}{3n} A_{k-1}(n,a) + \frac{2(k-1)a^2}{3n} A_{k-2}(n,a) .$$

This holds if $k \geq 2$. If $k = 1$, then we see from (5.27) that $B_1 = -\frac{1}{3n} A_0$. So if we define $A_{-1} = 0$, then (5.28) holds for $k \geq 1$.

Inserting (5.28) into (5.26), we have

$$(5.29)\quad A_k(n,a) = \frac{2(k-1)}{(3n)^2} [2(k-3)a^2 A_{k-4}(n,a) - (2k-5)A_{k-3}(n,a)],$$

and, on the other hand, if we put (5.26) into (5.28) we get

$$(5.30)\quad B_k(n,a) = \frac{2(k-1)a^2 \cdot 2(k-3)}{(3n)^2} B_{k-4}(n,a)$$

$$- \frac{(2k-1)2(k-2)}{(3n)^2} B_{k-3}(n,a).$$

In other words A_k and B_k both satisfy recursion formulae of the type

$$(5.31)\qquad D_k = \frac{1}{(3n)^2} (\lambda D_{k-4} + \mu D_{k-3})$$

where here and in what follows, λ and μ denote generic coefficients depending on k but not on n. In order to save writing we shall call all such coefficients λ not bothering to distinguish one from another by subscripts or the like.

Since A_k and B_k satisfy a relation such as (5.31), so too must C_k (see (5.24)),

$$C_k = \frac{1}{(3n)^2} (\lambda C_{k-4} + \mu C_{k-3})$$

$$= \frac{1}{(3n)^4} \left[\lambda C_{k-8} + f_2(C_{k-7}, C_{k-6})\right]$$

$$= \frac{1}{(3n)^6} \left[\lambda C_{k-12} + f_3(C_{k-11}, C_{k-10}, C_{k-9})\right],$$

or, in general

$$C_k = \frac{1}{(3n)^{2m}} \left[\lambda C_{k-4m} + f_m(C_{k-4m+1}; \cdots C_{k-3m})\right] , \tag{5.32}$$

where f_m is linear in all its arguments.

Now

$$A_1 = 0 \; ; \quad B_1 = \frac{\lambda}{3n} A_0 \; .$$

Hence

$$C_1 = \frac{\lambda}{3n} C_0 \; .$$

$$A_2 = \frac{\lambda}{3n} B_0 \; ; \quad B_2 = \frac{\lambda}{3n} A_0 \; .$$

Therefore, $C_2 = \frac{\lambda}{3n} C_0$.

$$A_3 = \frac{\lambda}{3n} B_1 = \frac{\lambda}{(3n)^2} A_0 \; ;$$

$$B_3 = \frac{\lambda}{3n} A_2 + \frac{\mu}{3n} A_1 = \frac{\lambda}{(3n)^2} B_0 \; .$$

Hence

$$C_3 = \frac{\lambda}{(3n)^2} C_0 \; .$$

From (4.32) it therefore follows that if $k = 4m$

$$(5.33)\quad C_{4m} = \frac{\lambda}{(3n)^{2m}} C_0 + o(n^{-2m})C_0 .$$

If $k = 4m+1$,

$$C_{4m+1} = \frac{\lambda}{(3n)^{2m}} C_1 + \frac{\lambda}{(3n)^{2m}} C_2 + o(n^{-2m-1})C_0 ,$$

or

$$(5.34)\quad C_{4m+1} = \frac{\lambda}{(3n)^{2m+1}} C_0 + o(n^{-2m-1})C_0 .$$

If $k = 4m+2$, then

$$C_{4m+2} = \frac{\lambda}{(3n)^{2m}} C_2 + \frac{\mu}{(3n)^{2m}} C_3 + \frac{\nu}{(3n)^{2m+2}} C_0 + o(n^{-2m-2})C_0 .$$

$$(5.35)\quad C_{4m+2} = \frac{\lambda}{(3n)^{2m+1}} C_0 + o(n^{-2m-1})C_0 .$$

If $k = 4m+3$, then

$$C_{4m+3} = \frac{\lambda}{(3n)^{2m}} C_3 + \frac{\mu}{(3n)^{2m+2}} C_0 + o(n^{-2m-2})C_0 .$$

$$(5.36)\quad C_{4m+3} = \frac{\lambda}{(3n)^{2m+2}} C_0 + o(n^{-2m-2})C_0 .$$

Collecting (5.33) - (5.36), we have

$$(5.37)\quad \begin{cases} C_k = O(n^{-\frac{k}{2}})C_0 & \text{if } k = 4m \\ C_k = O(n^{-\frac{k+1}{2}})C_0 & \text{if } k = 4m+1 \end{cases}$$

$$(5.37)\quad \begin{cases} C_k = O(n^{-\frac{k}{2}})C_0 & \text{if } k = 4m+2 \\ C_k = O(n^{-\frac{k+1}{2}})C_0 & \text{if } k = 4m+3\ , \end{cases}$$

or

$$(5.37')\quad \begin{cases} C_k = O(n^{-\frac{k}{2}})C_0 & \text{if } k \text{ is even,} \\ C_k = O(n^{-\frac{k+1}{2}})C_0 & \text{if } k \text{ is odd.} \end{cases}$$

It remains to estimate the remainder,

$$(5.38)\quad R_1(n,a) = \int_{L_+'} e^{n(3a^2w-w^3)}\, r_1(w^2,a)dw\ .$$

To do this we use (5.10). Since ϕ is analytic and of exponential order at ∞, so too is Λ_1. Just as in previous cases it therefore follows that

$$(5.39)\quad R_1(n,a) = \int_{L_+} e^{n(3a^2w-w^3)}\Lambda_{-1}(w,a)(a^2-w^2)^{K+1}dw$$

$$+ O(e^{-nq}) \qquad (q > 0).$$

For simplicity, assume that K+1 is even. (If K+1 is odd, only minor changes in the following argument are necessary.)

Integrating (5.39) by parts gives

$$(5.40)\quad R_1(n,a) = -\frac{1}{3n}\int_{L_+} e^{n(3a^2w-w^3)}\frac{d}{dw}[\Lambda_1 (a^2-w^2)^K]dw+O(e^{-nq}).$$

Now it is clear that we can continue to integrate by parts until we reach an integral in which (a^2-w^2) no longer appears, since each integration by parts decreases by one the highest power of (a^2-w^2) appearing. It is also clear that this will happen first when we apply all the $\frac{d}{dw}$'s occurring in the repeated integration by parts to the power of (a^2-w^2) left. If we apply some of the differentiations to the Λ_1, then we shall have to carry out more integrations by parts in order to eliminate the (a^2-w^2) from the integrand. These additional integrations by parts will increase the power of $\frac{1}{n}$ multiplying the resulting integrals. Since K+1 is assumed even, it is therefore evident from the treatment of the C_k integrals that

$$(5.41)\quad R_1(n,a) = O(n^{-\frac{K+1}{2}})\int_{L_+} e^{n(3a^2w-w^3)}\Lambda_1(w,a)dw\ .$$

Once again we use the fact that the part of the integral taken over L_+'' is $O(e^{-nq})$. This means that

$$R_1(n,a) = O(n^{-\frac{K+1}{2}})\int_{L_+'} e^{n(3a^2w-w^3)}\Lambda_1(w,a)dw\ .$$

But on L_+', $|\Lambda_1(w,a)| < \Lambda$. Hence

$$|R_1(n,a)| \leqq O(n^{-\frac{K+1}{2}}) \int_{L_+'} |e^{n(3a^2w-w^3)}dw| \quad ,$$

and therefore, surely

$$(5.42) \qquad |R_1(n,a)| \leqq O(n^{-\frac{K+1}{2}}) \int_{L_+} |e^{n(3a^2w-w^3)}dw| \quad .$$

Similarly, for

$$R_2(n,a) = \int_{L_+} e^{n(3a^2w-w^3)} w\, r_2(w^2,a)dw \quad ,$$

we have

$$(5.43) \qquad |R_2(n,a)| \leqq O(n^{-\frac{K+1}{2}}) \int_{L_+} |we^{n(3a^2w-w^3)}dw| \quad .$$

The estimates (5.42) and (5.43) differ slightly from those obtained for the individual termsof the series. Since, however,

$$|A_0(n,a)| = \left| \int_{L_+} e^{n(3a^2w-w^3)}dw \right| \quad ,$$

$$|A_0(w,a)| \leqq \int_{L_+} |e^{n(3a^2w-w^3)}dw| ;$$

likewise

$$|B_0(n,a)| \leqq \int_{L_+} |we^{n(3a^2w-w^3)}dw| \quad ;$$

therefore, if we put

$$C_0^* = \int_{L_+} |e^{n(3a^2w-w^3)}dw| + \int_{L_+} |we^{n(3a^2w-w^3)}dw| \ ,$$

then obviously

$$|R_i(n,a)| \leqq O(n^{-\frac{K+1}{2}})C_0^* \tag{5.44}$$

and (cf. (5.24))

$$|C_0(n,a)| \leqq M''C_0^* \quad . \tag{5.45}$$

Relations (5.37'), (5.44) and (5.45) establish the asymptotic character of the series (5.25). This, in turn, together with (5.11) and (5.16), establishes the validity of (5.23) as an asymptotic representation for F(n,a) and completes the theoretical part of the discussion.

§6. Calculation of the Terms in the Expansion

We turn next to the practical side of the problem and the first question which naturally arises is how to compute the terms in our asymptotic expansion (1.14). What have to be computed are the $c_k^{(i)}(a)$ and the A_k's and B_k's. According to (5.7)

$$(6.1) \qquad c_k^{(i)}(a) = \frac{(-1)^k}{k!}\, \phi_i^{(k)}(a^2,a)$$

where $\phi_i^{(k)}(a^2,a)$ denotes the k^{th} derivative of ϕ_i with respect to its first argument evaluated at a^2.

For the purpose of the applications it is somewhat more convenient to express the $c_k^{(i)}$ in terms of ϕ instead of ϕ_1 and ϕ_2. Since ϕ_1 and $w\phi_2$ are just the even and odd parts of ϕ, this is not hard:

$$(6.2) \qquad \phi_1(w^2,a) = \tfrac{1}{2}[\phi(w,a) + \phi(-w,a)]$$

$$(6.3) \qquad \phi_2(w^2,a) = \frac{1}{2w}[\phi(w,a) - \phi(-w,a)] \ .$$

Hence

$$(6.4) \qquad c_0^{(1)}(a) = \phi_1(a^2,a) = \tfrac{1}{2}[\phi(a,a) + \phi(-a,a)]$$

$$(6.5) \qquad c_0^{(2)}(a) = \phi_2(a^2,a) = \frac{1}{2a}[\phi(a,a) - \phi(-a,a)] \ .$$

The $c_k^{(i)}$ with $k > 0$ can be computed from (6.2) and (6.3) by

differentiating with respect to w^2 and then setting $w = a$.

It should be noted, however, that in the computation of the coefficients $c_k^{(i)}(a)$ by means of formulas (6.2)-(6.5) indeterminate expressions such as $\frac{0}{0}$ will always occur. These then have to be evaluated by using L'Hopital's rule once if $a \neq 0$ and twice if $a = 0$. That this is necessary should be no surprise, for it will be recalled that at the very outset when we looked for a method to resolve the difficulty of the coalescence of two branch points one of the key ideas was to make the numerator as well as the denominator of the expression for $\frac{dz}{dw}$ vanish and thereby render the expression indeterminate in the hope that it would then approach a finite non-zero limit when $a \to 0$. That this hope can be realized has been proved above and we therefore know that the coefficients can be computed by L'Hôpital's rule and will all be finite with the first one of them different from zero (some after the first might be zero, however, which is no contradiction and no harm).

As an illustration of these ideas, we shall compute $c_0^{(1)}(a)$, the first and most important coefficient. According to (1.12)

$$(6.6) \qquad \phi(\pm a, a) = \underline{\Phi}(z_{\pm})\frac{dz}{dw}\Big|_{\substack{w=\pm a \\ z=z_{\pm}}} .$$

We assume for simplicity that $\underline{\Phi}$ is regular and $\neq 0$ at $z_{\pm}$.

For the application we have in mind $\underline{\mp}$ is in fact a constant. We therefore only have to compute

$$\left.\frac{dz}{dw}\right|_{\substack{w=\pm a \\ z=z_\pm}} = \lim_{\substack{w\to\pm a \\ z\to z_\pm}} \frac{dz}{dw}$$

$$= \lim_{\substack{w\to\pm a \\ z\to z_\pm}} \frac{3(a^2-w^2)}{-f_z(z,a)} \qquad \text{(by (1.8))}$$

$$= \lim_{\substack{w\to\pm a \\ z\to z_\pm}} \frac{6w}{f_{zz}(z,a)\frac{dz}{dw}} \qquad \text{(by L'Hôpital's rule).}$$

Thus,

$$(6.7) \qquad \lim_{\substack{w\to\pm a \\ z\to z_\pm}} \frac{dz}{dw} = \sqrt{\frac{\pm 6a}{f_{zz}(z_\pm,a)}} \qquad \text{for } a \neq 0,$$

$$= \sqrt[3]{\frac{6}{f_{zzz}(z_0,a)}} \qquad \text{if } a = 0.$$

Therefore,

$$(6.8) \qquad \phi(\pm a,a) = \mp\Phi(z_\pm)\sqrt{\frac{\pm 6a}{f_{zz}(z_\pm,a)}} \ .$$

So, by (6.4),

$$(6.9) \qquad c_0^{(1)} = \tfrac{1}{2}\,\mp\Phi(z_+)\sqrt{\frac{6a}{f_{zz}(z_+,a)}} + \tfrac{1}{2}\,\mp\Phi(z_-)\sqrt{\frac{-6a}{f_{zz}(z_-,a)}} \ .$$

It remains to express the A_k's and B_k's in terms of some well known functions. Since, as remarked in the introduction, the recursion formulas (1.17) and (1.18) (or (5.26) and (5.28)) permit all the A_k's and B_k's with $k \geqq 1$ to be expressed in terms of A_0 and B_0, it is only necessary to express A_0 and B_0 in terms of known functions. This is done by putting

$$(6.10) \qquad \begin{cases} w = (3n)^{-1/3} & \text{and} \\ a = (3n)^{-1/3} z^{1/2} \end{cases}$$

in (1.15) (or (5.21)). Then

$$(6.11) \qquad A_0(n,a) = (3n)^{-1/3} \int_{L_+} e^{zt - \frac{1}{3}t^3}\, dt$$

or

$$(6.12) \qquad A_0(n,a) = \frac{2\pi i}{(3n)^{1/3}}\, Ai_+(z) \ ,$$

where $Ai_+(z)$ is the Airy integral defined by (1.20). Thus,

$$(6.13) \qquad A_0(n,a) = \frac{2\pi i}{(3n)^{1/3}}\, Ai_+[(3n)^{2/3} a^2] \ , \text{ which is (1.21).}$$

As for B_0,

$$(6.14) \qquad B_0(n,a) = \int_{L_+} w e^{n(3a^2 w - w^3)}\, dw \quad \text{(by (1.22))}$$

$$= \frac{1}{3n} \frac{\partial}{\partial a^2} \int_{L_+} e^{n(3a^2w-w^3)} dw$$

$$= \frac{1}{3n} \frac{\partial}{\partial a^2} A_0(n,a) ,$$

so that by (6.13)

$$(6.15) \qquad B_0(n,a) = \frac{2\pi i}{(3n)^{2/3}} Ai'_+[(3n)^{2/3} a^2] ,$$

which is (1.22)

To obtain (1.24), merely put

$$(6.16) \qquad t = e^{-\frac{i\pi}{3}} \tau$$

in (1.20). Then

$$(6.17) \qquad Ai(z) = \frac{e^{-\frac{i\pi}{3}}}{2\pi i} \int_{L_1} e^{e^{-\frac{i\pi}{3}} z\tau + \frac{1}{3}\tau^3} d\tau ,$$

whence by (1.23),

$$(6.18) \qquad Ai(z) = \frac{e^{-\frac{i\pi}{3}}}{2k} h_1(e^{-\frac{i\pi}{3}} z),$$

so that

$$(6.19) \qquad A_0(n,a) = \frac{\pi i \, e^{-\frac{i\pi}{3}}}{k(3n)^{1/3}} h_1(e^{-\frac{i\pi}{3}} z) ,$$

or

$$(6.20)\quad A_0(n,a) = \frac{\pi\, e^{\frac{i\pi}{6}}}{k(3n)^{1/3}}\, h_1[e^{-\frac{i\pi}{3}}(3n)^{\frac{2}{3}} a^2] ,$$

which is (1.24).

From (6.14) and (6.20), we then get (1.25):

$$(6.21)\quad B_0(n,a) = \frac{\pi\, e^{-\frac{i\pi}{6}}}{k(3n)^{2/3}}\, h_1'[e^{-\frac{i\pi}{3}}(3n)^{\frac{2}{3}} a^2] .$$

As mentioned in the introduction, formulas (1.24) and (1.25) are the more useful ones for actual numerical computations. Also important is the observation that although only a^3 is specified by (1.9), there is no latitude in the choice of a, because $\sqrt[3]{a}$ must be chosen so as to be consistent with the choice of cube roots made on pages 37-39. In general, it is best to examine each individula case on its own merits, taking into account the particular function $f(z,a)$ involved. This point will be further illustrated in §8.

§7. Parameters Not Independent

The formula (1.14) and the special case of it (1.28) are good for all sufficiently small values of α (and therewith a) and sufficiently large values of n and, as remarked many times before, retain their validity even as $\alpha \to 0$. They were derived under the assumption that α and n were independent parameters. If, however, α depends upon n in such a way that $\alpha \to 0$ as $n \to \infty$, then it is clear that (1.14) and (1.28) are also applicable in this case since it is but a special case of the more general one which has been treated above. For example, if

$$\alpha = \alpha(n) \tag{7.1}$$

where

$$\lim_{n \to \infty} \alpha(n) = 0, \tag{7.2}$$

then a, being a function of α, is also a function of n:

$$a = a(n) \tag{7.3}$$

such that

$$\lim_{n \to \infty} a(n) = 0 , \tag{7.4}$$

and (1.28) becomes

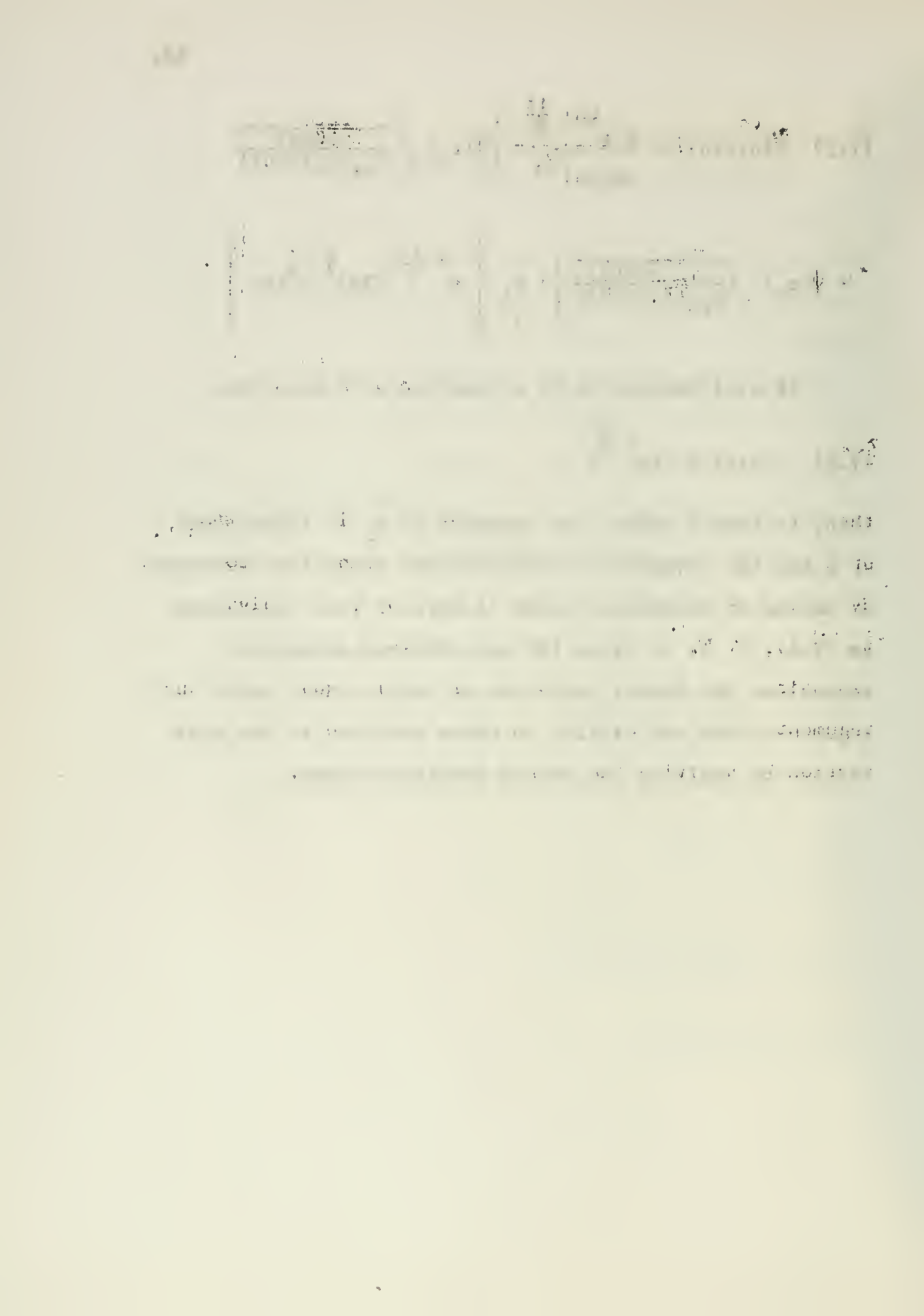
$$(7.5)\quad F[n,a(n)] \sim \frac{\pi\, e^{bn+\frac{i\pi}{6}}}{2k(3n)^{1/3}} \left\{ \Phi(z_+) \sqrt{\frac{6a(n)}{f_{zz}[z_+,a(n)]}} \right.$$

$$\left. + \Phi(z_-) \sqrt{\frac{-6a(n)}{f_{zz}[z_-,a(n)]}} \right\} \cdot h_1 \left[e^{-\frac{i\pi}{3}} (3n)^{\frac{2}{3}} a^2(n) \right] .$$

If a(n) happens to be a function of n such that

$$(7.6)\quad a(n) = O(n^{-\frac{1}{3}}) ,$$

then, to lowest order, the argument of h_1 is independent of n and the asymptotic representation simplifies somewhat. By making an assumption which is more or less equivalent to (7.6), F. W. J. Olver [8] has obtained asymptotic expansions for Bessel functions of nearly equal order and argument which are similar to those obtained in the next section by applying the method developed above.

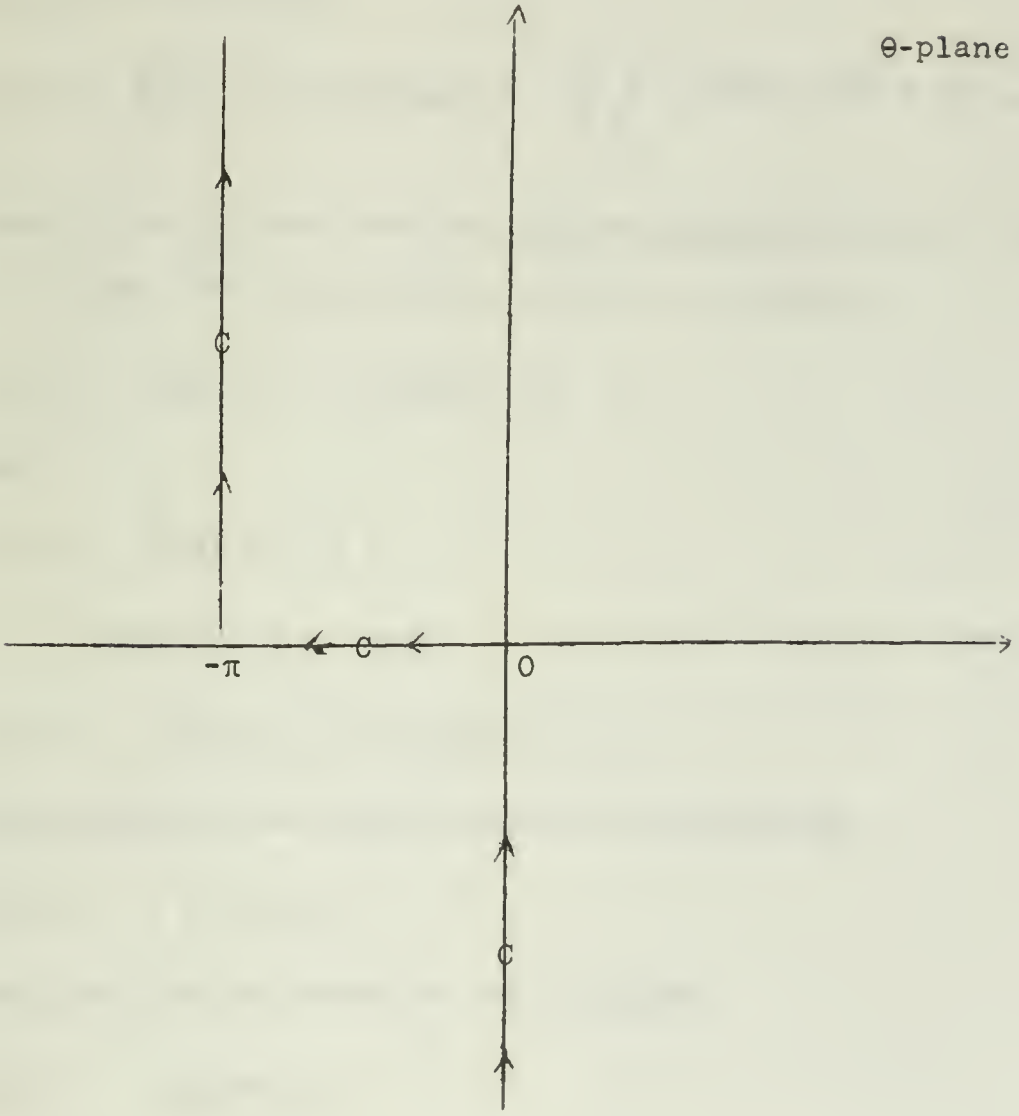


§8. Application to Hankel Functions

The Hankel functions [2],

$$(8.1) \qquad H_n^{(1)}(z) = -\frac{1}{\pi}\int_C e^{-i(z\sin\theta - n\theta)}\,d\theta\,,$$

where C is the path shown below:



have been the subject of numerous investigations, a number of which have had as their object the determination of the asymptotic behavior of $H_n^{(1)}(z)$. It is found that the principal cases of interest are those in which both the order, n, and the argument, z, become large and, in fact, at the same rate. More precisely, the important cases are those in which [11]

$$(8.2) \qquad H_n^{(1)}(z) = H_n^{(1)}(nr) = -\frac{1}{\pi}\int_C e^{-in(r\sin\theta - \theta)}\,d\theta$$

where r is a fixed positive number independent of n. This is of the form (1.1) with z called θ, α called r,

$$(8.3) \qquad f(\theta,r) = i(r\sin\theta - \theta)$$

and

$$(8.4) \qquad \phi(\theta) = -\frac{1}{\pi}\,.$$

From (8.3) we have

$$(8.5) \qquad f'(\theta,r) = i(r\cos\theta - 1)\,.$$

Consequently, the branch points of the mapping

$$(8.6) \qquad \zeta = -f(\theta,r)$$

are given by the roots of the equation

$$(8.7) \qquad \cos\theta = \frac{1}{r}\,.$$

To determine their order, we have to look at

$$f''(\theta,r) = -ir\sin\theta \,. \tag{8.8}$$

From (8.7) and (8.8) it is clear that all branch points are simple if $r \neq 1$ but at least double if $r = 1$. In fact, since

$$f'''(\theta,r) = -ir\cos\theta \,, \tag{8.9}$$

they are precisely double if $r = 1$.

We therefore have a situation in which <u>all</u> branch points change their order as the parameter, in this case r, tends to a certain value (one). We have already seen, however, that if we perform the mapping (8.6), then the dominant terms in the asymptotic expansion of $H_n^{(1)}(nr)$ will be contributed by those branch points in the θ-plane whose images in the ζ-plane lie closest on the right to the curve Γ in the ζ-plane which is the image of C under (8.6). It is well known from the theory of Bessel functions that it is the branch point $\theta = \theta_0$ closest to the origin from which the main part of the asymptotic expansion arises. According to the discussion of sections 1-6 it will therefore suffice to remove the point θ_0 as a branch point. Naturally the branch point $-\theta_L$ with which it coalesces must also be removed. We therefore take for the points $\theta_{\pm}$

$$\theta_{\pm} = \pm\theta_0 \tag{8.10}$$

where θ_0 is the root of (8.7) having smallest absolute value.

Since

$$r\cos\theta_0 = 1 \,, \tag{8.11}$$

it follows from (8.3) that

$$\begin{aligned} f(\theta_{\pm},r) &= \pm i(r\sin\theta_0 - \theta_0) = \pm i(\tan\theta_0 - \theta_0) \\ &= \pm i\left(\sqrt{r^2-1} - \tan^{-1}\sqrt{r^2-1}\right) \\ &= \pm\int_1^r \sqrt{1-\frac{1}{\rho^2}}\,d\rho \quad . \end{aligned} \tag{8.12}$$

Hence, according to (1.9) and (1.10)

$$a^3 = -\frac{i}{2}(\tan\theta_0 - \theta_0) = -\frac{i}{2}\left(\sqrt{r^2-1} - \tan^{-1}\sqrt{r^2-1}\right) \tag{8.13}$$

and

$$b = 0, \tag{8.14}$$

while according to (8.8) and (8.12)

$$f''(\pm\theta_0,r) = \mp i\sqrt{r^2-1} \quad . \tag{8.15}$$

To determine which cube root to take in (8.13), note that for a = 0, (1.8) becomes (cf.(8.3))

$$-w^3 = \frac{-i\theta^3}{6} + \ldots \,,$$

or

$$w = \left(\frac{i}{6}\right)^{1/3}\theta + \ldots \quad .$$

For $a \neq 0$,

$$3a^2w - w^3 = i(r-1)\theta + \dots ,$$

or

$$w = i\left(\frac{r-1}{3a^2}\right)\theta + \dots \quad .$$

We therefore have to choose the cube root in a such that

$$i\,\frac{r-1}{3a^2} \longrightarrow \left(\frac{i}{6}\right)^{1/3} \quad \text{(where we take } i^{1/3} = e^{\frac{i\pi}{6}}\text{)},$$

i.e., since r is real, such that $a^2 i^{1/3}$ is pure imaginary and has imaginary part of the same sign as r-1. This means

$$a = \frac{e^{\frac{7\pi i}{6}}}{2^{1/3}}\left(\sqrt{r^2-1} - \tan^{-1}\sqrt{r^2-1}\right)^{1/3} ,$$

if $r > 1$, and

$$a = \frac{e^{\frac{2\pi i}{3}}}{2^{1/3}}\left(\sqrt{1-r^2} - \tanh^{-1}\sqrt{1-r^2}\right)^{1/3}$$

if $r < 1$. Therefore, formula (1.28) tells us that if $r > 1$, then

$$(8.16)\quad H_n^{(1)}(nr) \sim \frac{-i}{k(3n)^{1/3}}\sqrt{\frac{-6\left(\sqrt{r^2-1} - \tan^{-1}\sqrt{r^2-1}\right)^{1/3}}{2^{1/3}\sqrt{r^2-1}}}\cdot h_1(z),$$

where

$$z = \frac{(3n)^{2/3}\left(\sqrt{r^2-1} - \tan^{-1}\sqrt{r^2-1}\right)^{2/3}}{2^{2/3}} \quad .$$

If $r < 1$, then

$$(8.17)\quad H_n^{(1)}(nr) \sim \frac{k}{k(3n)^{1/3}}\sqrt{\frac{6\left(\sqrt{1-r^2} - \tanh^{-1}\sqrt{1-r^2}\right)^{1/3}}{2^{1/3}\sqrt{1-r^2}}}\cdot h_1(z),$$

where

$$z = \frac{(3n)^{2/3}\left(\sqrt{1-r^2} - \tanh^{-1}\sqrt{1-r^2}\right)^{2/3}}{-2^{2/3}} \quad .$$

For r = 1, we have to use (1.27). This gives

$$H_n^{(1)}(n) \sim \frac{6^{1/3} h_1(0)}{k(3n)^{1/3}} . \tag{8.18}$$

As mentioned previously, the ordinary saddle point method gives rise to three different formulae, each valid for a different range of r. These are the following (cf., Watson [11] (pp. 243-4)):

$$H_n^{(1)}(nr) \sim \frac{e^{in(\sqrt{r^2-1} - \tan^{-1}\sqrt{r^2-1}) - \frac{1}{4}\pi i}}{\sqrt{\frac{1}{2}n\pi\sqrt{r^2-1}}} \qquad (r > 1); \tag{8.19}$$

$$H_n^{(1)}(nr) \sim \frac{e^{n(\sqrt{1-r^2} - \tanh^{-1}\sqrt{1-r^2})}}{\sqrt{2\pi n\sqrt{1-r^2}}} \tag{8.20}$$

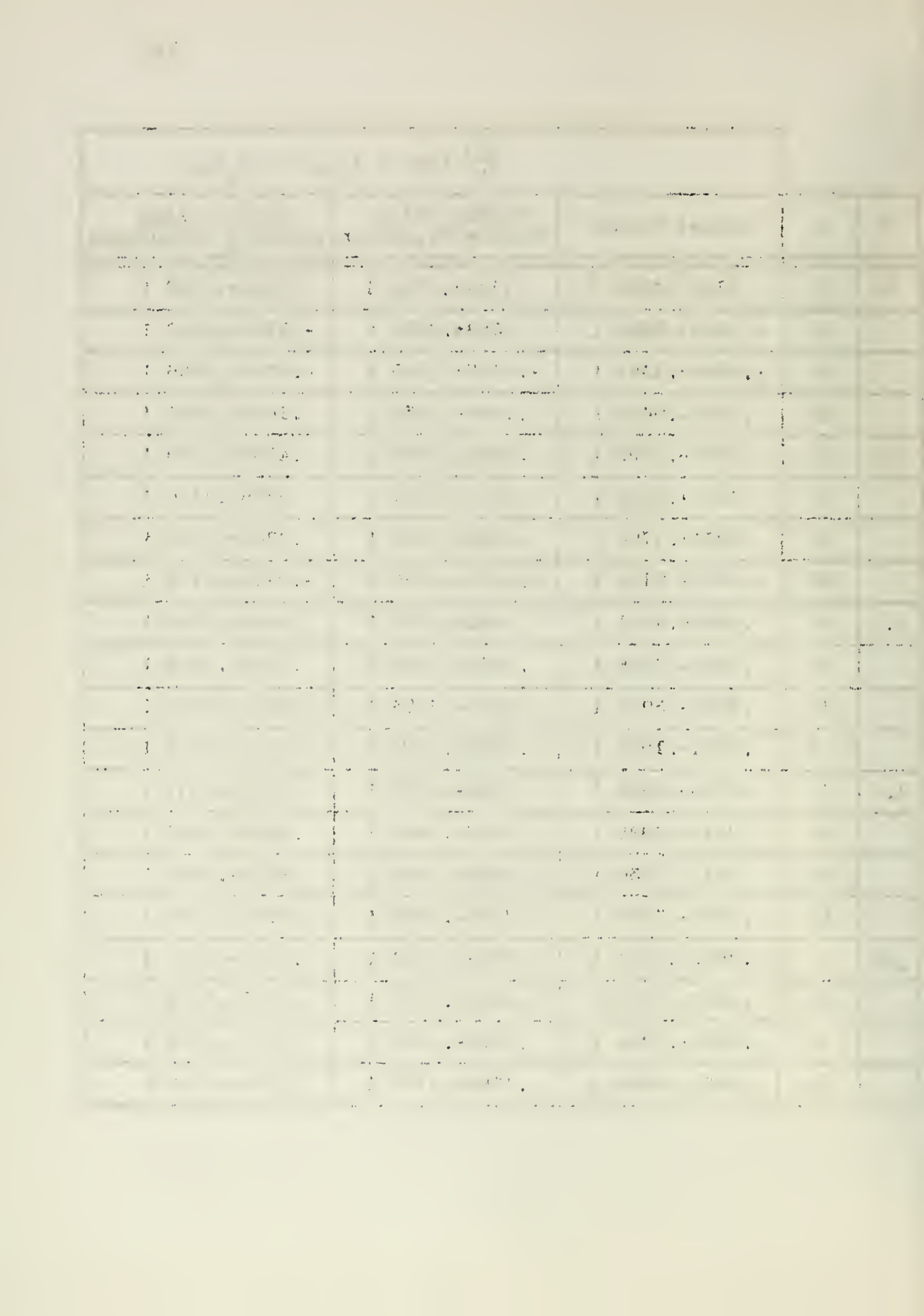
$$- \frac{ie^{n(\tanh^{-1}\sqrt{1-r^2} - \sqrt{1-r^2})}}{\sqrt{\frac{1}{2}\pi n\sqrt{1-r^2}}} \qquad (r < 1);$$

$$H_n^{(1)}(n) \sim \frac{\Gamma(\frac{1}{3})}{2^{2/3}\pi n^{1/3}} \left[\frac{1}{3^{1/6}} - i3^{1/3} \right] \tag{8.21}$$

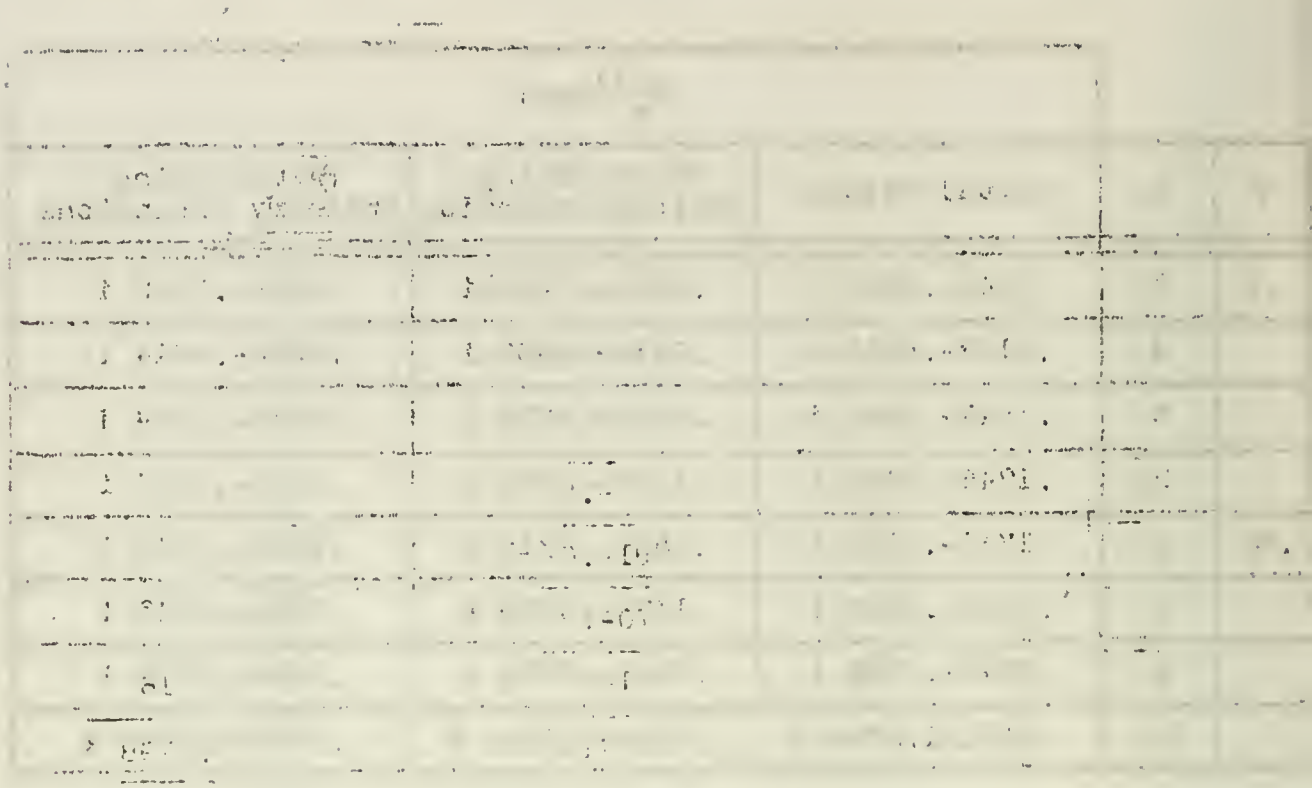
Formula (8.21) is, in fact, identical with (8.18), as is readily seen by using the definition (1.23) of $h_1(z)$ in (8.18).

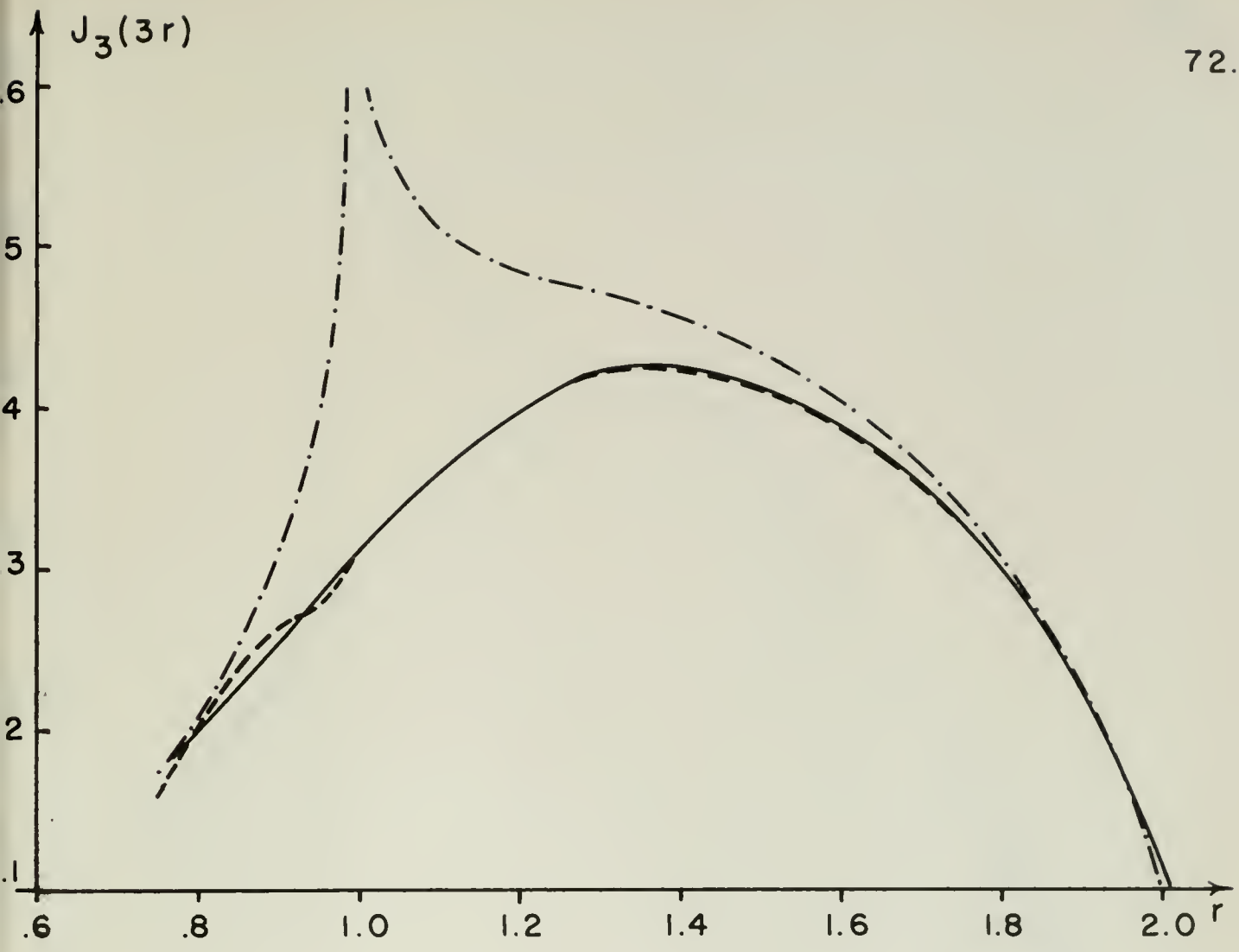
The following table and graphs offer a comparison among the actual values of $H_n^{(1)}(nr)$, those computed by using the uniform expansion ((8.16)-(8.18)) and those computed by using the ordinary expansion ((8.19)-(8.21)):

		$H_n^{(1)}(nr) = J_n(nr) + iY_n(nr)$		
r	n	Actual Values	Values Using Uniform Expansion	Values Using Ordinary Expansions
2	3	.1147+.3282 i	.1144+.3310 i	.1040+.3342 i
	4	-.1053+.2829 i	-.1058+.2818 i	-.1133+.2812 i
	5	-.2340+.1354 i	-.2337+.1345 i	-.2376+.1306 i
	10	.1864-.0438 i	.1856-.0435 i	.1871-.0419 i
1.5	3	.4247-.0090 i	.4240-.0074 i	.4352+.0199 i
	4	.3576+.0983 i	.3570+.0991 i	.3578+.1196 i
	5	.2834+.1754 i	.2826+.1756 i	.2786+.1904 i
	10	-.0900+.2199 i	-.0882+.2162 i	-.0959+.2185 i
1.1	3	.3587-.4291 i	.3595-.4273 i	.5206-.4382 i
	4	.3364-.3698 i	.3369-.3688 i	.4615-.3665 i
	5	.3209-.3260 i	.3211-.3253 i	.4220-.3159 i
	10	.2804-.1983 i	.2803-.1979 i	.3271-.1786 i
1.0	3	.3090-.5385 i	.3101-.5371 i	.3101-.5371 i
	4	.2811-.4889 i	.2817-.4880 i	.2817-.4880 i
	5	.2611-.4536 i	.2615-.4530 i	.2615-.4530 i
	10	.2074-.3598 i	.2076-.3596 i	.2076-.3596 i
.95	3	.2818-.5969 i	.2754-.6093 i	.3923-.8371 i
	4	.2507-.5522 i	.2561-.5433 i	.3361-.7328 i
	5	.2280-.5214 i	.2426-.4955 i	.2974-.6623 i
	10	.1661-.4439 i	.1789-.4177 i	.1992-.4945 i



		$H_n^{(1)}(nr)$		
r	n	Actual Values	Values Using Uniform Expansion	Values Using Ordinary Expansions
.9	3	.2540-.6600 i	.2643-.6415 i	.3068-.7402 i
	4	.2197-.6215 i	.2118-.6386 i	.2575-.6614 i
	5	.1947-.5963 i	.2061-.5733 i	.2233-.6104 i
	10	.1246-.5454 i	.1133-.5798 i	.1350-.5046 i
.75	3	.1711-.9101 i	.1591-.6219 i	.1728-.7719 i
	4	.1320-.9167 i	.1370-.8970 i	.1309-.7643 i
	5	.1046-.9445 i	.0641-.9634 i	.1024-.7816 i
	10	.0421-1.2769 i	.0348-1.0191 i	.0370-1.0798 i

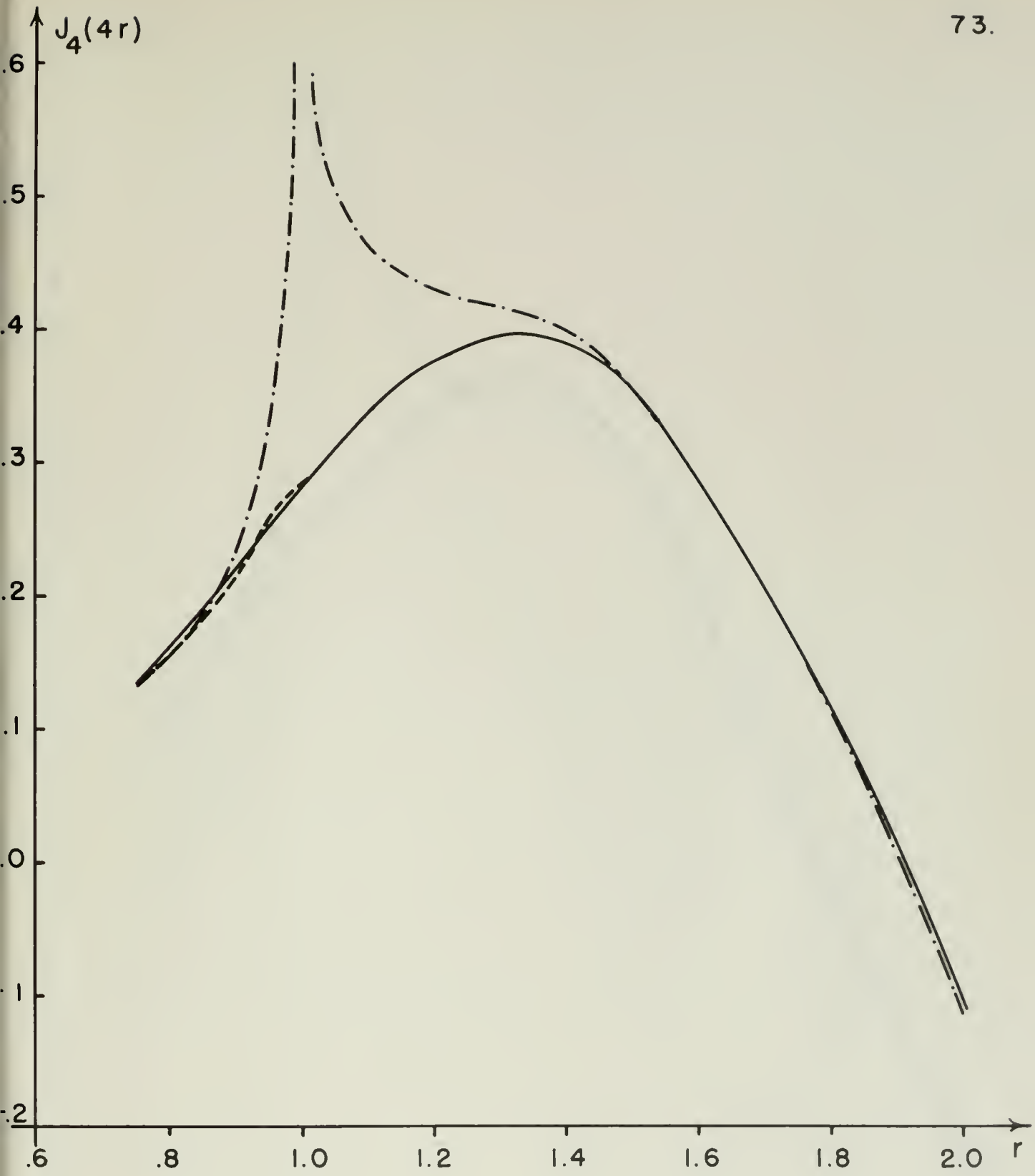




——— Actual value

------ Uniform expansion

—·—·— Ordinary expansion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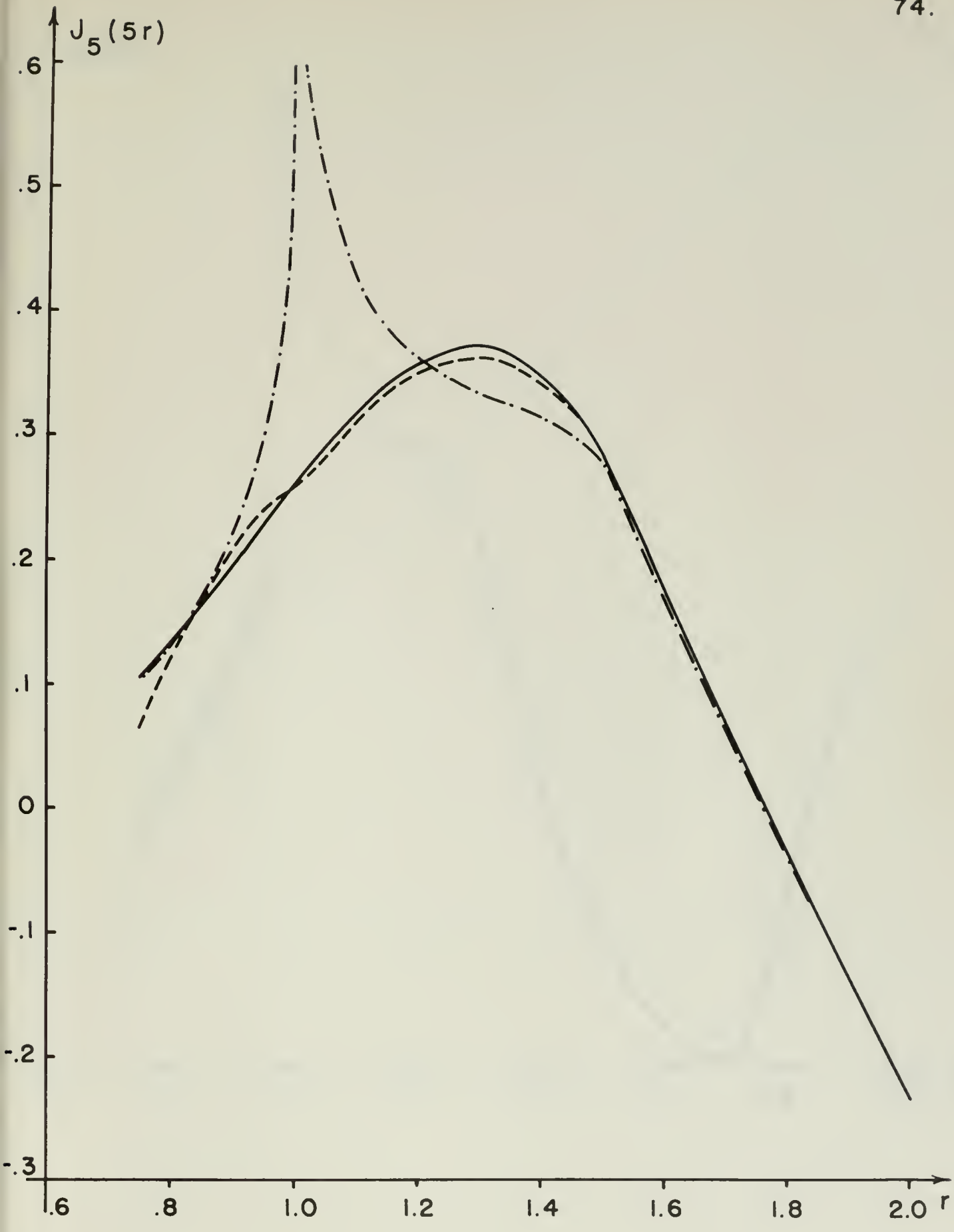
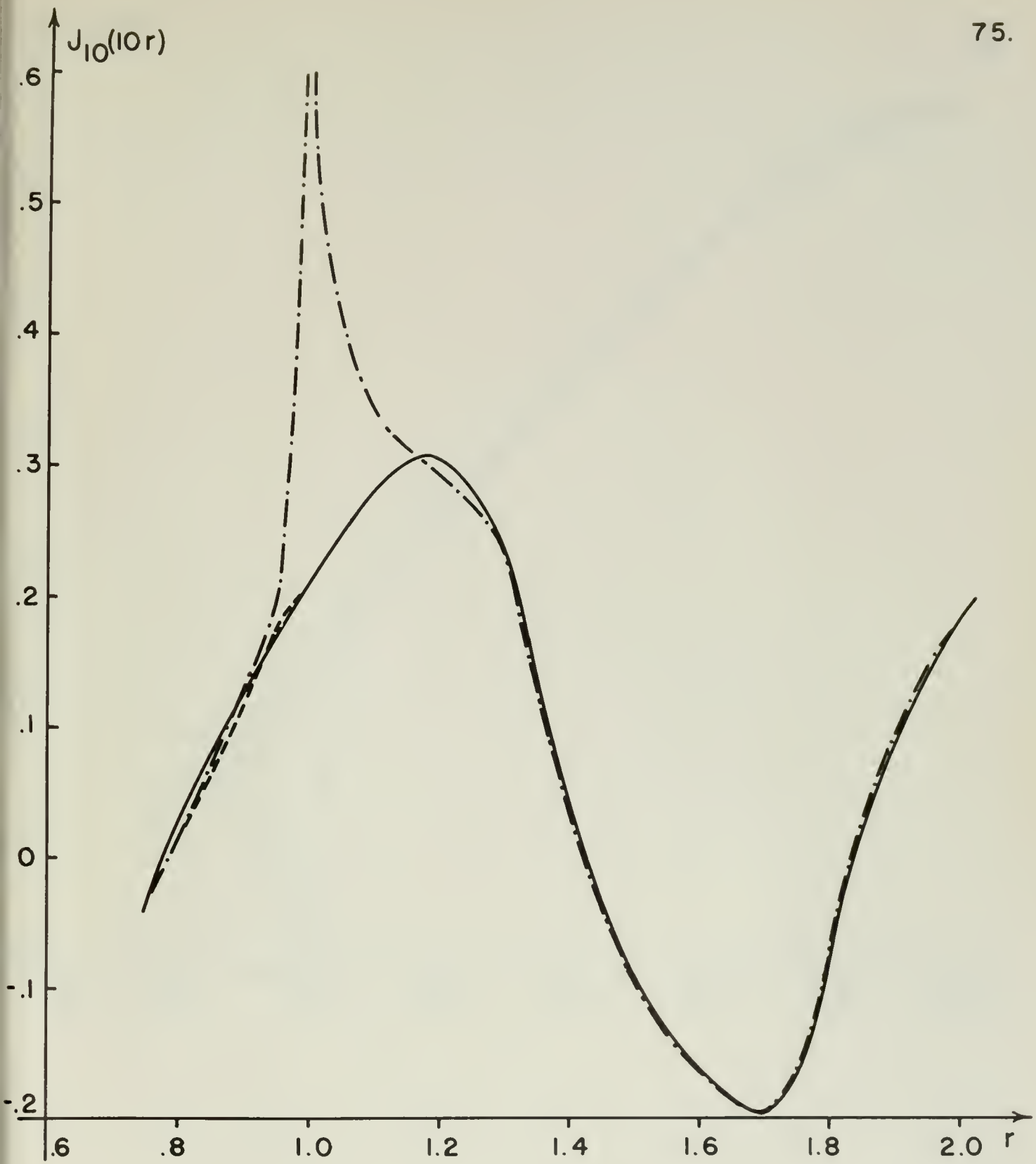
.5
.4
.3
.2
.1
.0
-.1
-.2
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
r


$J_5(5r)$
.6
.5
.4
.3
.2
.1
0
-.1
-.2
-.3
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
r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.5
.4
.3
.2
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0
-.1
-.2
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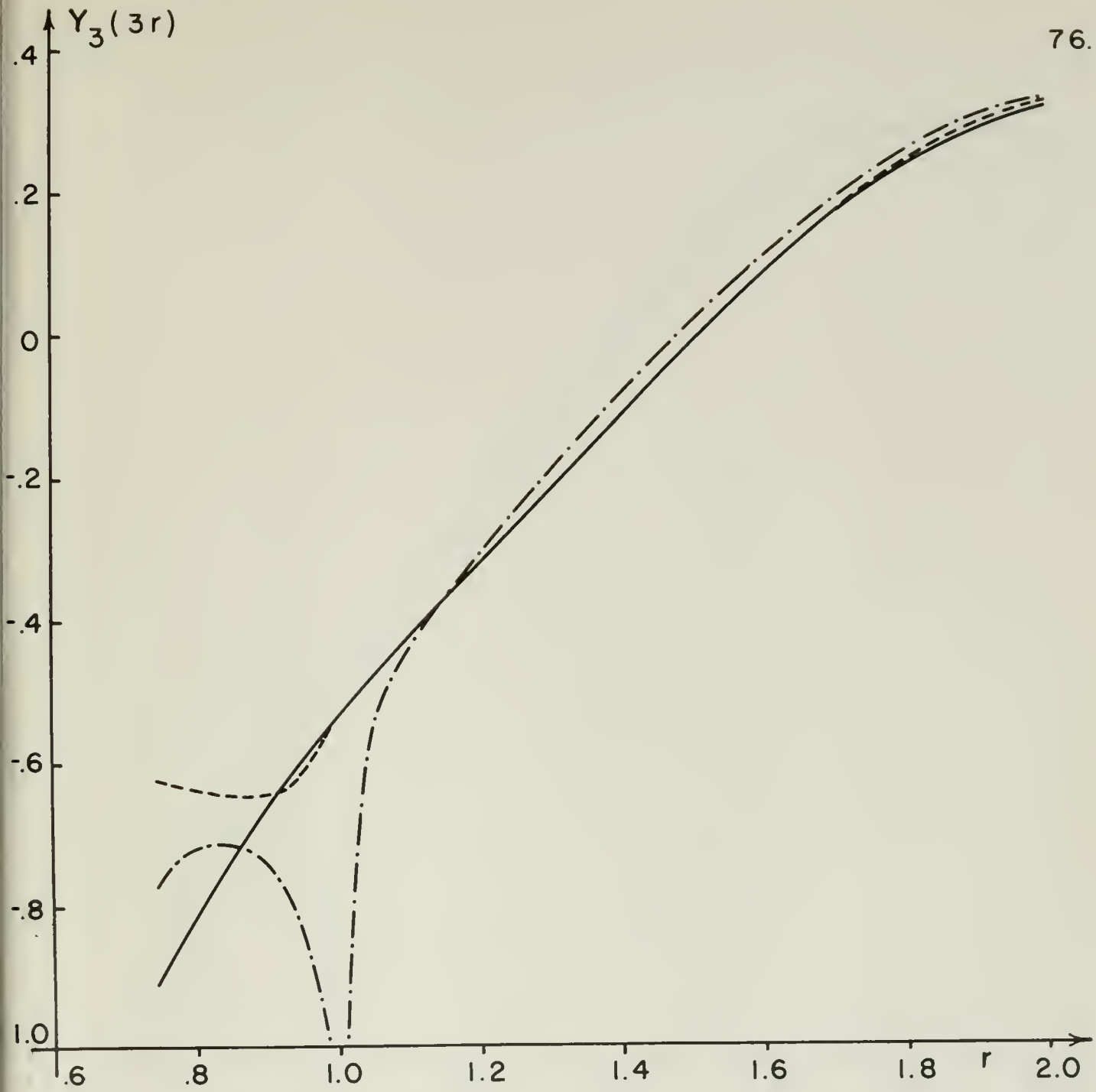
1.2
1.4
1.6
1.8
2.0
r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.2
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-.4
-.6
-.8
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.8
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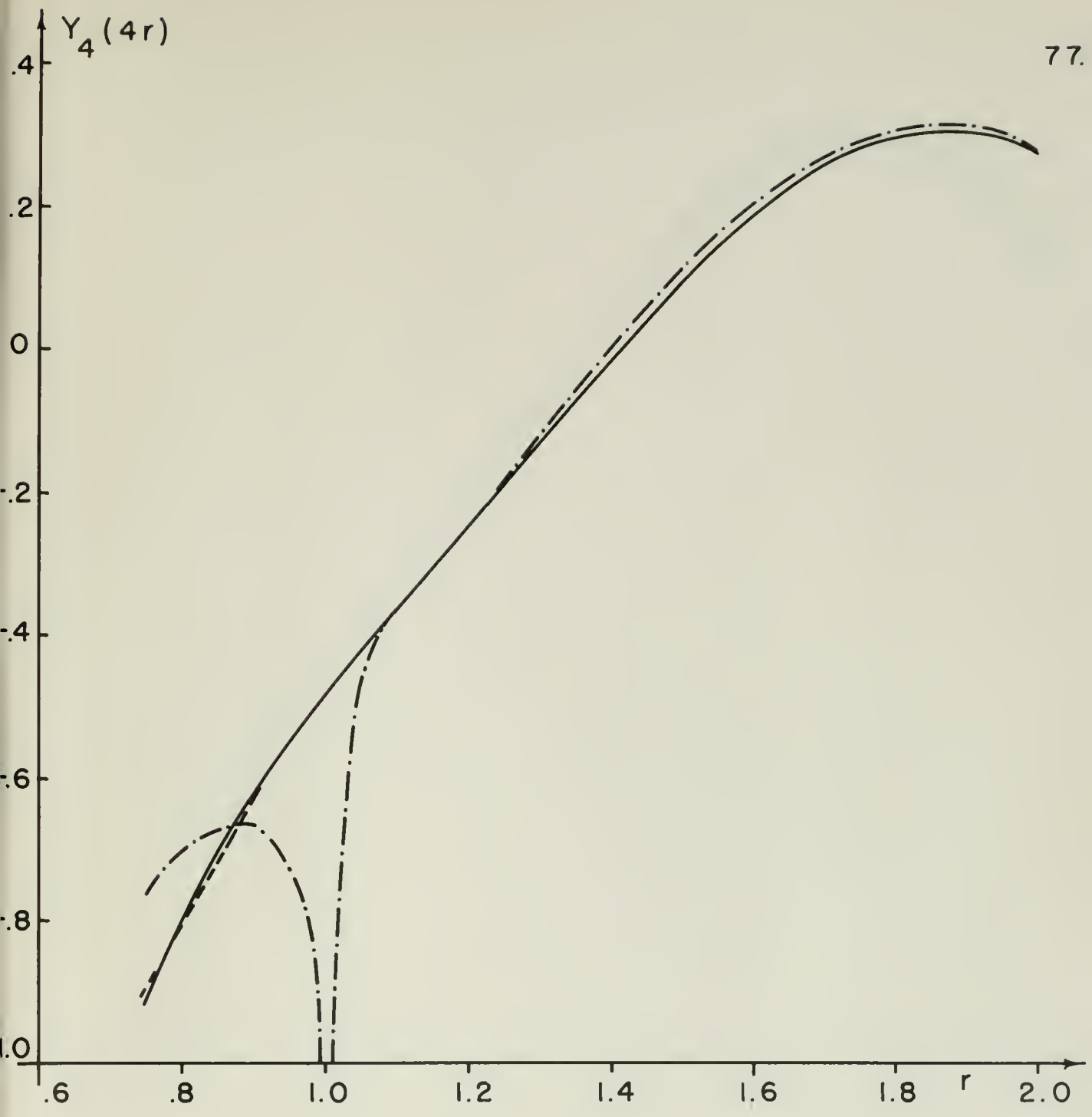
1.2
1.4
1.6
1.8
r
2.0


$Y_4(4r)$
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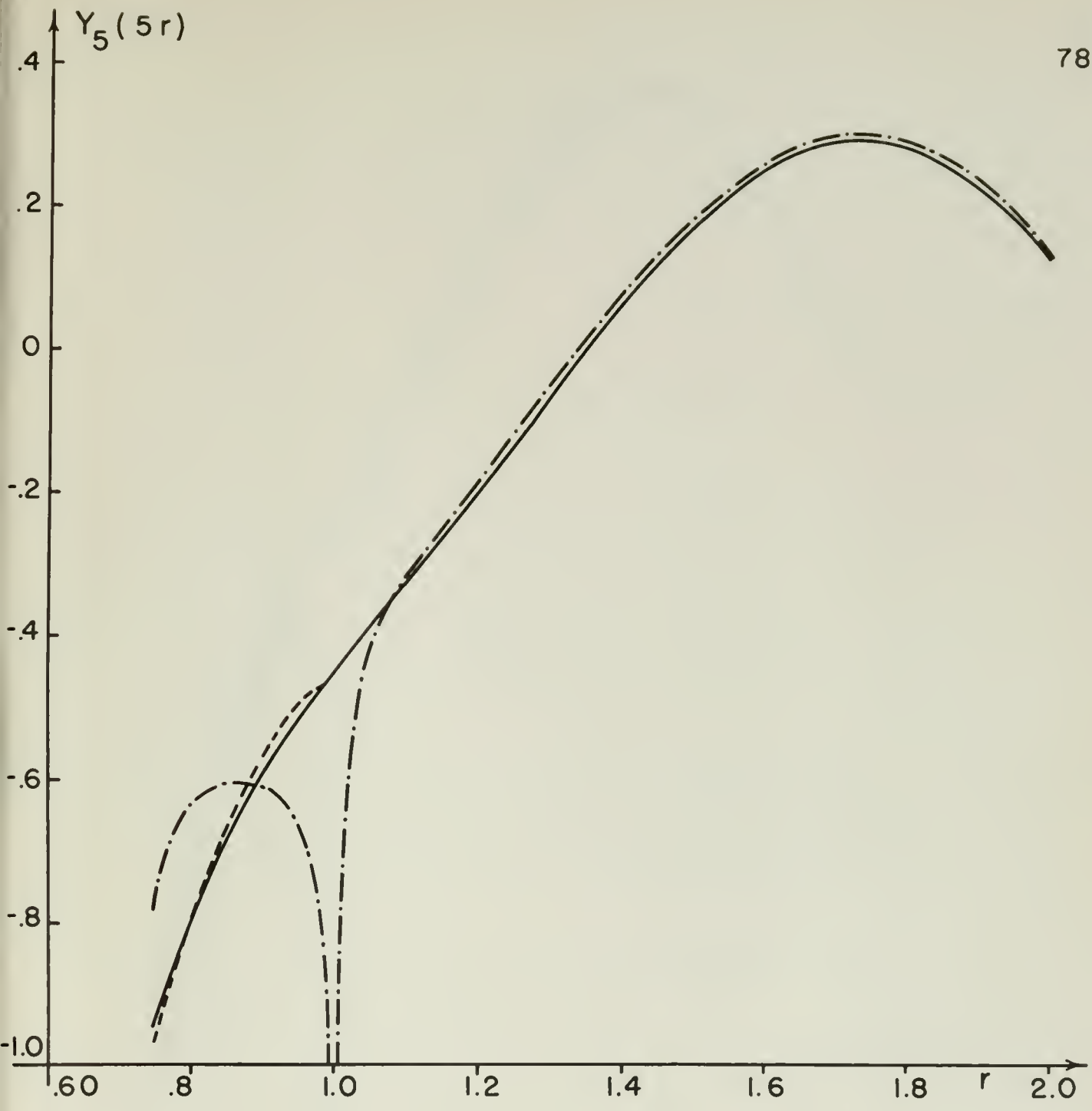
.2
0
-.2
-.4
-.6
-.8
-1.0
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
r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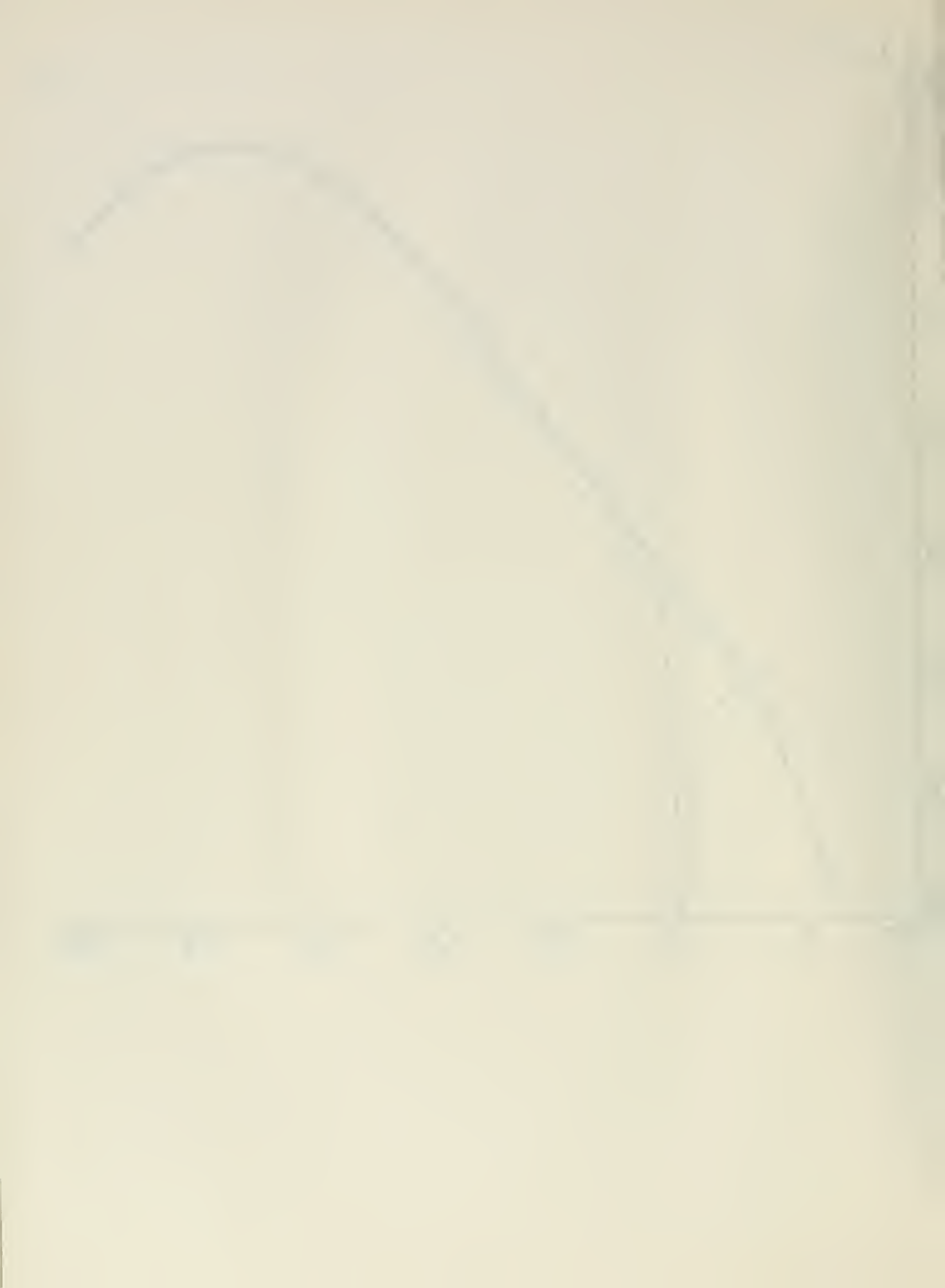
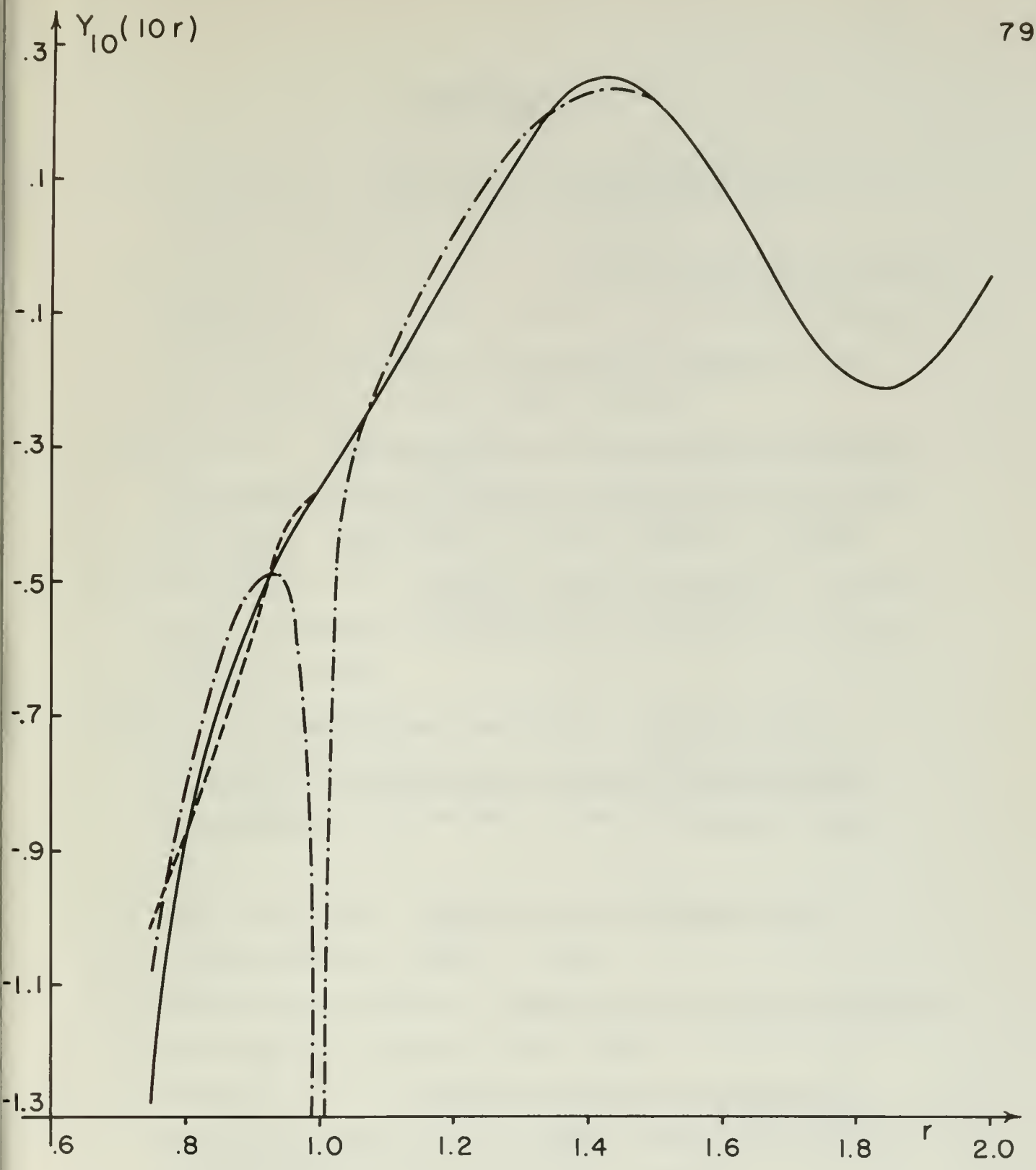
-.8
-1.0
.60
.8
1.0
1.2
1.4
1.6
1.8
r
2.0




$Y_{10}(10r)$
.3
.1
-.1
-.3
-.5
-.7
-.9
-1.1
-1.3
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
r

BIBLIOGRAPHY

[1] Bieberbach, L. Lehrbuch der Funktionentheorie, Bd. 1. Chelsea, New York, 1945, pp. 190-192.

[2] Courant, R. and Hilbert, D. Methoden der Mathematischen Physik, Bd. 1. Springer, Berlin, 1931, p. 407. English translation, Methods of Mathematical Physics, Vol. 1 Interscience, New York, 1953, p. 468.

[3] Debye, P. Näherungsformeln für die Zylinderfunktionen für grosse Werte des Arguments und unbeschränkte Werte des Index. Math. Ann., Vol. 67, 1909, pp. 535-558.

[4] Friedrichs, K. O. Special Topics in Analysis. Lecture notes, Institute of Mathematical Sciences, N. Y. Univ., 1953-4, p. B-67.

[5] Harvard Computational Laboratory. Tables of the Modified Hankel Functions of Order $\frac{1}{3}$ and of their Derivatives. Harvard Univ. Press; Cambridge, Mass., 1945.

[6] Hurwitz-Courant. Lehrbuch der Funktionentheorie. Springer, Berlin, 1929, p. 138.

[7] Jeffreys and Jeffreys. Methods of Mathematical Physics. Cambridge Univ. Press, 1946, p. 476.

[8] Olver, F. W. J. Some New Asymptotic Expansions for Bessel Functions of Large Orders. Proc. of Cambridge Philos. Soc., Vol. 48, 1952, pp. 414-427.

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